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Faculty of Education

Josip Juraj Strossmayer
University of Osijek
School of Applied Mathematics
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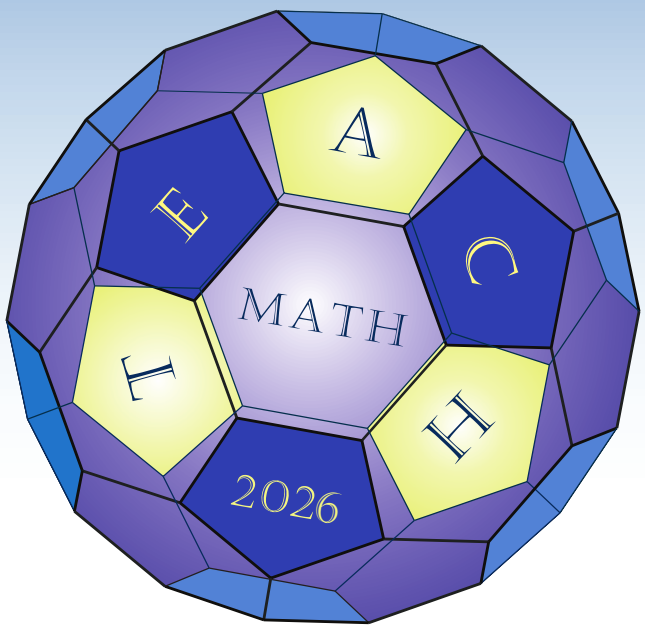

DEVELOPMENTS IN MATHEMATICS EDUCATION: MATHEMATICAL THINKING, LEARNING AND TECHNOLOGY

Editors:

Zdenka Kolar-Begović
Ružica Kolar-Šuper
Ana Katalenić
Ljerka Jukić Matić

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Josip Juraj Strossmayer University of Osijek
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DEVELOPMENTS IN MATHEMATICS EDUCATION: MATHEMATICAL THINKING, LEARNING AND TECHNOLOGY

RAZVOJ MATEMATIČKOG OBRAZOVANJA: MATEMATIČKO MIŠLJENJE, UČENJE I TEHNOLOGIJA

monograph

Editors:

Zdenka Kolar-Begović
Ružica Kolar-Šuper
Ana Katalenić
Ljerka Jukić Matić

Osijek, 2026

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A word from the Editorial Board

Every generation of mathematics educators has been offered a new instrument and told that it would change how mathematics is learned. The abacus and the counting frame gave way to the printed textbook, which for centuries defined what mathematics was worth knowing and in what order. Manipulatives promised to make abstraction tangible; the overhead projector promised to make the teacher's reasoning visible; the pocket calculator promised to release students from computation so that they might finally think. Dynamic geometry software promised the same for construction, spreadsheets for modelling, and the interactive whiteboard for classroom discourse as a whole. Most recently, digital textbooks, educational games and generative artificial intelligence have arrived with claims of similar magnitude. The pattern that accompanies these arrivals is remarkably stable: enthusiasm, rapid adoption, mixed empirical results, and a quieter period in which the profession works out what the instrument is actually good for.

It would be easy to read this history as a series of disappointments. We read it differently. What the record shows is not that tools fail, but that tools do not teach. An instrument changes what is possible in a classroom; it does not by itself change what happens there. The calculator did not make students mathematically literate, nor did it make them innumerate — its effect depended almost entirely on the tasks it was embedded in and on what teachers asked students to do with the results. The same has proved true of every technology since. This is not an argument for pessimism about technology. It is a claim about where the educational work actually resides.

That work resides in mathematical thinking, and it is worth being precise about what we mean by the term. Pólya (1945) described mathematical thinking as a repertoire of heuristics — understanding the problem, devising a plan, carrying it out, looking back — that could be taught and learned rather than merely possessed. Freudenthal (1973) reframed mathematics as an activity rather than a body of results, insisting that students should reinvent mathematical structures under guidance rather than receive them ready-made. More recent theoretical work has sharpened the picture further. Duval (1995) showed that geometrical understanding requires the coordination of distinct modes of apprehending a figure, and that perception alone never suffices. Sfard (2008) characterised mathematics as a discourse with its own vocabulary, visual mediators and endorsed narratives, into which learners must be inducted. What these accounts share is the recognition that mathematical thinking is neither a by-product of exposure to content nor an automatic consequence of engagement with an interesting object.

This has direct implications for how we should understand technology in mathematics education. Vérillon and Rabardel (1995) drew the essential distinction between an artefact and an instrument: an artefact is a material or symbolic object, whereas an instrument is that object together with the schemes a user has

developed for employing it. The transformation of the first into the second, instrumental genesis, is neither immediate nor guaranteed, and it is precisely the part of the process that a technology cannot perform on its own. Trouche (2004) demonstrated how much this process depends on deliberate orchestration by the teacher, and Drijvers et al. (2010) showed that identical software in different classrooms produces different mathematical activity depending on how its use is staged. The instrument, in other words, is made in the classroom rather than delivered to it.

Empirical syntheses converge on the same conclusion from a different direction. Reviews of technology in mathematics education report modest average effects concealing very large variation, with the variation attributable to instructional design rather than to the technology itself (Hillmayr et al., 2020). Studies of digital game-based learning find consistent gains in motivation and engagement alongside far less consistent gains in conceptual understanding — the affective effect is close to reliable, the cognitive effect is contingent (Byun & Joung, 2018). Hattie's (2009) broad synthesis places feedback, clarity of goals and the quality of instructional design well above the presence of any particular resource. The message is not that these tools are ineffective. It is that their effectiveness is conditional, and that the conditions are pedagogical.

Two of those conditions recur often enough to deserve naming. The first is the alignment of task, scaffold and assessment. A tool supports mathematical thinking when what students are asked to do with it, the guidance they receive while doing it, and the evidence gathered about what they have understood all point in the same direction; where these come apart, engagement and learning come apart with them. Assessment deserves particular emphasis here, because it is the point at which the alignment most often fails quietly. A student who reaches a correct answer by a visual or structural route, in a classroom where the expected form of a solution is a written procedure, may find that route not marked wrong but simply not recognised — and will draw the appropriate conclusion about what counts as mathematics. The second condition is the mediating role of the teacher. Across settings and instruments, the decisive variable is not whether students encounter a rich object but whether someone asks the questions that move them from noticing to reasoning, from manipulation to justification. This is the least automatable part of mathematics education and, on the present evidence, the least dispensable.

None of this argues for caution about technology, and we would not want it read that way. Digital environments make accessible forms of mathematical activity that were previously out of reach: dynamic exploration of invariance, immediate feedback on conjectures, representations that can be transformed and compared in real time, and modes of assessment that reveal student thinking rather than merely scoring it. The task is not to weigh technology against thinking, as though they were competing claims on limited classroom time. The task is to design environments in which the former serves the latter.

This is the sense in which the three terms of this volume's title belong together. Mathematical thinking, learning and technology are not three parallel research areas that happen to share a field. They stand in a definite relation: technology is valuable in mathematics education to the degree that it is made to serve the development of

mathematical thinking, and it is the design of learning — the tasks, the questions, the sequencing, the feedback — that determines whether it does. The contributions gathered here examine that relation from many directions and at many educational levels, and they are, we believe, unusually candid about where it holds and where it does not. We hope readers will find in these pages both encouragement and thoughtful reflection.

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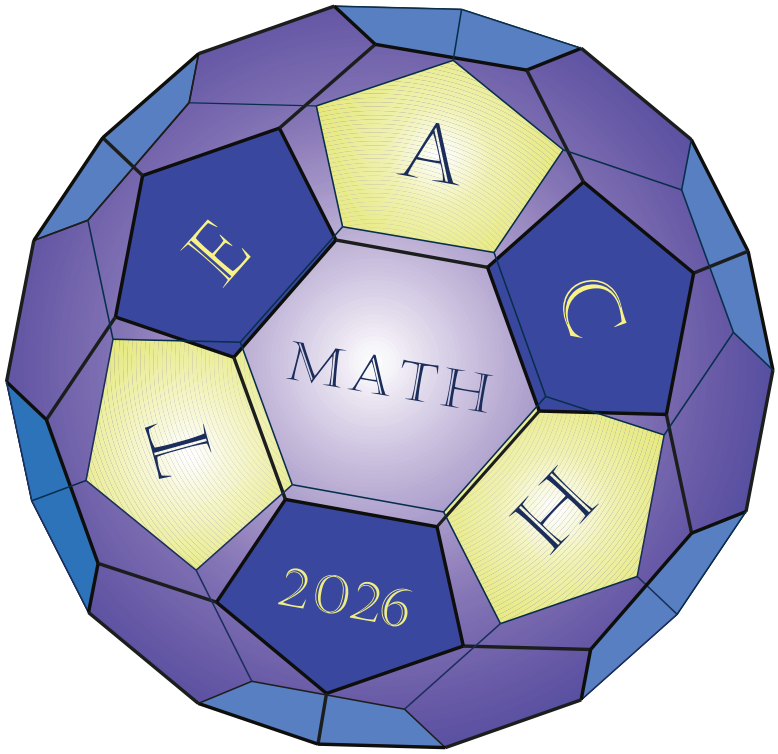
DEVELOPMENTS IN MATHEMATICS EDUCATION: MATHEMATICAL THINKING, LEARNING AND TECHNOLOGY

monograph



Preface

Mathematics education is shaped today by three forces that are easy to name separately and difficult to hold together: our growing understanding of how mathematical thinking develops, the accumulated evidence about how learning is best supported, and the rapid expansion of digital environments in which these processes now take place. Examining how these three relate to one another is the main thrust of this monograph, *Developments in Mathematics Education: Mathematical Thinking, Learning and Technology*. It brings together studies from early primary education to university teacher preparation, from Croatia, Slovenia, Serbia, Bosnia and Herzegovina, Kosovo, Hungary and Romania, and from research traditions ranging from classroom action research to systematic review and pure geometry.

The first chapter, *Mathematical Cognition, Problem Solving, and Student Attitudes in Mathematics Education*, concerns what students actually do when they are invited to think mathematically. The first paper analyses the problems sixth-graders themselves pose when learning fractions, finding that while most are mathematically sound, they overwhelmingly favour familiar everyday contexts over conceptual reasoning — a finding with clear consequences for how such activities should be framed. The second paper examines the relationship between contextual tasks and students' attitudes toward mathematics, showing that satisfaction with such tasks, rather than the sheer number completed, predicts perceived usefulness and motivation. The third paper takes up Krutetskii's and Sharma's distinction between quantitative and qualitative mathematical personality and traces it in the written solutions of pre-service primary teachers and of pupils with a special interest in mathematics. The quantitative orientation predominates in every group and on both tasks, and the characteristic error is a well-executed algorithm answering a question the task had not asked; a documented instance in which a pupil's correct visual solution was scored as "mere drawing" shows how readily a qualitative route is devalued when it does not take the expected procedural form. The final paper in this chapter demonstrates how Simpson's paradox, a standard topic in descriptive statistics, becomes intuitive when reformulated geometrically, and argues that what appears paradoxical to the untrained eye is not paradoxical at all to a mind trained in mathematics.

The second chapter, *Innovative Teaching Approaches, Representations, and Inclusion in Mathematics Education*, addresses the resources through which mathematical ideas are made accessible. The first paper traces seventy years of visual representations of division in primary textbooks from the region, documenting a gradual shift from procedural to relational treatments and showing how the choice of visual mediator privileges particular ways of understanding. The second paper applies Duval's theory of cognitive apprehension to lessons designed for seventh-graders, and finds that students remain at the level of perception unless the teacher's

questioning moves them toward discursive reasoning. The third paper reports an experiment with the Poly-Universe game in which prospective and secondary school students designed their own mathematical activities, with results that were creative but revealed no advantage for the more experienced group. The final paper examines the role of teaching assistants in inclusive mathematics instruction, where a strong commitment to student autonomy coexists with a unanimous call for resources and training that are not currently available.

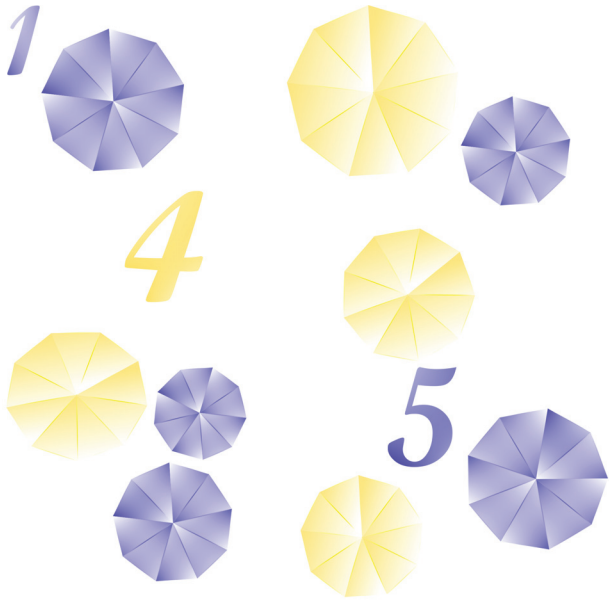
The third chapter, *Digital Learning Environments, Game-Based Learning, and Assessment in Mathematics Education*, takes up technology directly, and does so without easy optimism. The first paper compares how Croatian and Finnish secondary students experienced a lesson built around a digital geometry game, reporting greater enjoyment and markedly less frustration than in regular lessons, alongside criticism that the game resembled a quiz. The second paper systematically reviews research on Minecraft in mathematics and computer science education and reaches a conclusion that deserves emphasis: motivational gains are consistent, while cognitive gains appear only where task, scaffold and assessment are deliberately aligned. The third paper asks who the automated feedback in digital textbooks is actually intended for, and finds that teachers, receiving only summative information, turn to external tools to learn what their students understand.

The fourth chapter, *Research, Teacher Development, and Interdisciplinary Perspectives*, widens the frame. It opens with a retrospective on twenty years of the Mathematics and Children colloquium and its nine monographs, marking the tenth anniversary of the gathering in Osijek and reflecting on how the discipline has established itself in this region. The second paper analyses the teaching practice diaries of prospective mathematics teachers, finding that problem solving — the very heart of mathematical activity — is where they feel least secure and, tellingly, what they least often reflect upon. The third paper systematically reviews research on transitions in STEM education and notes, among other findings, the complete absence of such research in the Croatian context. The volume closes with a paper in which the author returns to his own work from 1983–84, explains the error it contained, and shows how that error connects to the quantum dots recognised by the 2023 Nobel Prize in Chemistry — a reminder that in mathematics a mistake carefully pursued is sometimes worth more than the result originally intended.

Taken together, these contributions point to a conclusion that no single chapter could establish on its own. Manipulatives, textbooks, games, digital environments and automated feedback all broaden the range of mathematical activity that can take place in the classroom, and none of them substitutes for the questions a teacher asks, the tasks a designer sets, or the reflection a learner is guided into. We offer this collection to teachers, researchers and those who shape educational policy, in the hope that it will prove useful in the work of making technology serve thinking.

Zdenka Kolar-Begović
Ružica Kolar-Šuper
Ana Katalenić
Ljerka Jukić Matić

1.
Mathematical Cognition, Problem Solving, and Student Attitudes in Mathematics Education



Exploring Students' Understanding of Fractions Through Problem Posing and Problem Solving

Blerina Tafolli Avdylaj[1] and Tatjana Hodnik[2*]

[1]Faculty of Education, University of Prishtina "Hasan Prishtina", Kosovo
[2]Faculty of Education, University of Ljubljana, Slovenia

Abstract. Developing students' competence in posing and solving mathematical problems is an important aspect of learning as it can support a deeper understanding of mathematical concepts. Therefore, it is important that students are repeatedly exposed to activities that promote problem posing and problem solving. Such activities should not only be treated as additional exercises or as final tasks at the end of a lesson, but they should be systematically and meaningfully integrated into all phases of the teaching process. This study examines sixth-grade students' competence in posing fraction problems during instruction that combines problem posing and problem solving methods. In addition, the study examines the structure and nature of the fraction problems posed by the students. Through a qualitative analysis of 66 formulated problems (posed by twenty students), these were categorized as *solvable* and *unsolvable*. Of the 48 solvable problems (73%), most real-life application problems (85%) involved fractions used as operators in familiar contexts. Only 15% of the problems were conceptual in nature and required reasoning, comparisons or explanations of relationships between fractions. These results suggest that, although students can pose mathematically solvable problems, these problems are often contextual (real-life application) and solved with minimal procedural steps, rather than encouraging deeper conceptual exploration of mathematical ideas. All solvable problems were then categorized into two groups: *structured* (81%) and *free problems* (19%). The low number of unsolvable problems (27%) suggests that students are able to pose solvable problems, but it also highlights the need to improve students' skills in structuring problems to ensure coherence with the mathematical concepts used. These findings emphasize the need to design teaching and learning processes that expose students to conceptual models of problem posing and solving that support the use of different representations of mathematical concepts as well as problems that promote reasoning. Another implication of this study is that teachers should be provided with didactic support and

*Corresponding author.

clear guidance to promote a deeper and more mathematically reflective approach to problem posing in the classroom. It is worth noting (and we are very aware of this in our research findings) that students can sometimes be confused about the teacher's expectations; they do not know what the teacher expects unless the teacher makes this clear. In other circumstances, however, reallife problems may be desirable, and students may be praised for posing them.

Keywords: fractions, problem posing, conceptual understanding, real-life applications, structured and free problems

1. Introduction

Among the mathematics teacher community, it is widely recognized that the typical students' comment on fractions is, "They are too difficult, which makes it harder to understand." The abstract nature of the fraction concept and the requirement for various representations, including numerical, visual, and verbal forms, provide significant challenges for students (Lamon, 2012; Birgin & Eryılmaz, 2025; Manfreda Kolar et al., 2018). Fraction competence is important for students' mathematics achievement beyond elementary school as well as in advanced mathematical algebra topics and proportional reasoning (Bailey et al., 2012). Therefore, investigating effective teaching methods to enhance students' understanding and achievement of fractions is a research area of interest. Effective mathematics teaching emphasizes fostering the development of procedural knowledge based on strong conceptual understanding. Students should learn to execute mathematical procedures and understand the fundamental concepts that make these procedures effective (Jukić Matić et al., 2020). Two teaching methods that have gained interest in this context are problem posing and problem solving; both are acknowledged for their capacity to improve students' mathematical performance (Cai & Hwang, 2020). Problem posing is fundamentally rooted in inquiry, which provides students with the opportunity to discuss and analyze a variety of problems created and presented by their peers. Studies show that problem posing fosters students' reflection on their understanding of mathematical concepts and improves their capacity to link ideas (Singer et al., 2013). Furthermore, it has been demonstrated to enhance students' attitudes towards mathematics by making the learning process more participatory and meaningful (Cai et al., 2012). On the other hand, problem solving focuses on finding appropriate strategies for solving mathematical problems. It entails logical reasoning, knowledge application, and dedication in solution discovery (Pólya, 1945). Problem solving is considered a fundamental aspect of mathematics education, as it provides students with critical skills for addressing real-life challenges and fosters an environment for growth (Schoenfeld, 1992). Pólya (1981) posits that four stages constitute the process of solving mathematical problems: 1) understanding the problem, 2) devising a problem solving plan, 3) executing a problem solving plan, and 4) reassessing the answer. Pólya's theory has been adapted into the general definition of problem solving and is widely employed in academic literature. Pólya's heuristic not only emphasizes the importance of

prior experiences and knowledge but also conveys these concepts in a concise, digestible, and teachable manner. This heuristic has become a common, if not the most popular, tool for teaching and learning problem solving (Liljedahl et al., 2016). Consequently, problem solving refers to the process of using mathematical concepts, techniques, and methods to analyze, understand, and find solutions to mathematical problems, with the primary objective of empowering students to utilize mathematical knowledge and critical thinking abilities to ascertain the correct answer and understand the reasoning behind the solution.

The integration of problem posing and problem solving in the teaching of fractions has the potential for addressing the challenges that students face in learning fractions. By encouraging students to generate and evaluate problems that incorporate fraction concepts, problem posing can facilitate the development of a more advanced understanding of fractions. In the same way, problem solving can offer students the chance to implement their knowledge in a variety of contexts, thereby strengthening their conceptual and procedural understanding. Therefore, this paper investigates the impact of combining problem posing and problem solving teaching methods on students' academic achievement in fractions. By investigating the potential benefits of these instructional methods, the research aim is to offer insights into effective teaching practices for enhancing mathematical performance in this fundamental area.

1.1. Research problem and research purpose

Previous research has often examined problem posing and problem solving separately, while studies investigating their combined use in fraction instruction remain limited. The purpose of this study is to examine the types of fraction problems posed by sixth-grade students during instruction that combines problem posing and problem solving teaching methods, with particular attention to the nature and structure of these problems. By addressing these aspects, this study aims to better understand the educational benefits of problem posing and problem solving methods, thereby contributing to the advancement of evidence-based practices in mathematics education and supporting teachers seeking to improve their students' performance in fractions. All of these issues are addressed through the following research questions:

1. *What kind of problems were posed by sixth-grade students while problem posing and problem solving teaching methods were applied?*
2. *What is the nature and structure of problems posed by the students (conceptual, real-life applications; free, semi-structured, and structured)?*

2. Literature review

2.1. Problem posing

In academic discourse, a singular, universally accepted definition of the concept of problem posing remains elusive, as various scholars have approached the subject from multifarious perspectives, depending on their respective research domains. Although attempts have been made to propose a new definition, most researchers have employed or adapted the two principal definitions posited by Silver (1994) and Stoyanova and Ellerton (1996). Silver (1994), a pioneer in this field, delineates problem posing as the generation of new problems and the reformulation of existing problems, which may occur before, during, or after the solution of a given problem. He associates these processes with Pólya's heuristic stages for problem solving. "Before" denotes the recreating of an existing problem in specific ways to render it more accessible to the solution. "During" signifies the transformation of a given problem statement into a new version that becomes the focus of solving, whereas "after" implies the generation of similar problems to those that students have already solved.

Stoyanova and Ellerton (1996, p. 518), in their paper on conducting a framework for students' problem posing in mathematics, have defined problem posing as "the process by which, based on mathematical experience, students use personal interpretations of concrete situations and formulate them as meaningful mathematical problems". Furthermore, they proposed a framework based on the idea that every problem posing situation can be categorized as free, semi-structured, or structured. A *free problem posing* situation refers to when students are asked to pose problems from a given situation and some instruction can be given to provoke a special activity. A *semi-structured* problem posing situation can involve a structured problem given by a picture, calculation, equation, or inequality and asking the students to pose questions or problems based on that information. Restating a problem based on its solution, improving the characteristics of a written solution, posing problems with different task formats are referred to as *structured problem posing.* More recent studies have examined how the type of problem posing situation influences the problems that learners generate. Silber and Cai (2017), for example, compared free and structured problem posing and found that the conditions under which problems are posed can influence how closely participants attend to the mathematical relationships involved in a task. In research focusing specifically on fractions, Martínez and Blanco (2021) examined the problems posed by 11- to 12-year-old students under different forms of support and analyzed both the types of fraction problems generated and whether students were able to solve the problems they had posed. These studies highlight the relevance of considering both the problem posing situation and the mathematical relationships represented in the problems students pose.

The field of problem posing is relatively new and has yet to be thoroughly developed or defined. Some scholars have focused on the cognitive processes involved in problem posing, while others have explored the educational implications of teaching problem posing abilities (Papadopoulos et al., 2022). While Yao et al.

(2021) suggest that future studies should focus on how teachers use problem posing as a teaching method to develop students' conceptual understanding, Cai and Hwang (2023) argue that engaging students in problem posing leads to deeper understanding and more flexible thinking. Furthermore, studies have shown that when problem posing is meaningfully integrated into instruction, it can enhance students' engagement and support students in developing autonomy in learning mathematics (Li et al., 2022). These contributions demonstrate how the field of problem posing is evolving into a more systematic and pedagogic method that should be integrated into classroom practice, rather than being seen merely as a supplementary ability, but as a core component of mathematical learning and reasoning.

2.2. Fractions

Recent research has increasingly emphasized the role of problem posing as an integral part of mathematics teaching and learning rather than as a supplementary activity (Li et al., 2022). Understanding fractions is a crucial step in mastering rational numbers, enabling students to have a simpler "mathematical journey" in middle and high school. Given that the concept of fractions is fundamental and widely utilized in school mathematics, emphasis on its acquisition is essential. Both procedural and conceptual skills are essential for students' success in fractions because the differences between mathematical performance are caused by differences in these skills (Lamon, 2012; Hiebert, 2013). A larger correlation exists between conceptual knowledge and students' overall cognitive abilities than between procedural knowledge and cognitive abilities (Lenz & Wittmann, 2020). Research indicates that students demonstrate a greater proficiency in procedural knowledge compared to conceptual understanding of fractions (Hallett et al., 2010; Lenz & Wittmann, 2020). For instance, students can execute a procedural computation to demonstrate the equivalence of two fractions more swiftly than employing a visual representation for the same purpose (Hallett et al., 2010). This reliance on procedures signifies a deficiency in conceptual understanding and underscores the need for approaches that integrate procedural abilities with the conceptual development of the fractions concept. The authors Xie and Masingila (2017) advocate for inquiry based and student-centered methods, including problem solving and problem posing to enhance fraction learning. These methods encourage students to explore fraction relationships, reason, and build their understanding rather than relying solely on algorithmic procedures. Toluk-Uçar (2009) engaged pre-service teachers in the experimental classes to pose fractions problems. Then, a discussion was raised on the appropriateness of the posed word problems, focusing on the justifications for their solvability and suitability for primary school students' thinking level. The discussion focused on why the problems were (or were not) appropriate, and participants were asked to support their arguments by using different representation systems, such as fraction bars, pattern blocks, number lines, or pictures, to justify their solutions or reasoning. The role of the teacher was to moderate the discussion by not providing a correct answer to posed problems but giving assumptions and provoking questions to encourage discussion. The findings indicate that problem posing helped pre-service teachers better understand

fractions and remember memorized rules and problem posing had a positive impact on pre-service teachers' views about what it means to know mathematics (Toluk-Uçar, 2009). Studies of fraction problem posing have also identified difficulties related to the formulation and mathematical coherence of the problems students pose. Özçakır Sümen (2021), in analyzing fourth-grade students' fraction problem posing, considered dimensions such as clarity, mathematical accuracy, complexity, and consistency with the given conditions. Difficulties may also become visible when quantities, units, or fraction relationships are not coordinated appropriately within the context of a posed problem. This is important when examining not only whether a problem is solvable, but also the specific difficulties that appear in its formulation.

In the mathematics education literature, it is well known that the concept of fractions is not unique but consists of several interrelated meanings. Different researchers have identified several common interpretations of fractions, including fractions as part-whole, as ratios, as operators, and as numbers on a number line (Manfreda Kolar et al., 2018). These representations of fractions can be challenging for students and often fail to develop all of them simultaneously. Given that this study addresses the posing and solving of problems involving fractions, it is essential to highlight that studies discussing the different interpretations of fractions provide a theoretical foundation for understanding how various instructional strategies, including problem posing, can support the development of fraction knowledge.

3. Methodology

3.1. Participants

This research involves twenty sixth-grade students from a public school in Kosovo. The sample selection is intentional, as sixth-grade students are at a pivotal point in their mathematical development, and their understanding of fractions is both a fundamental learning objective and a prerequisite for understanding further complex mathematical concepts. The sample size, although modest, is suitable for an action study design, facilitating an in-depth examination of individual and collective learning processes (Creswell & Creswell, 2018). The instructional activities during the intervention were implemented by the classroom teacher, who facilitated the teaching process. The teacher, however, was not considered a participant in the study, as no data were collected from her, nor were her attitudes, reflections, or professional perspectives included in the analysis.

3.2. Research methods

The study employs a *qualitative research approach* within an action research context to examine students' responses to problem posing and problem solving activities in the teaching of fractions. The classroom teacher implemented the action

research cycle, whereas the researchers designed the instructional materials, provided lesson plans, and analyzed the collected data. The researchers did not participate directly in the instruction or classroom observation, following a non-participant action research model. Action research is appropriate for examining complex phenomena in education. It facilitates the simple integration of qualitative and quantitative data to examine the multifaceted character of the learning process (Creswell & Creswell, 2018). By concentrating on a single classroom situation, the study identifies the challenges, achievements, and differences in students' responses to the problem posing and problem solving methods.

3.3. Instruments

Students' written work was collected throughout the intervention phase for the analysis of the posed problems. A total of 66 posed problems were collected. During the intervention, students engaged in problem posing in pairs/individually across three different situations: free, semi-structured, and structured problem posing.

3.4. Procedure

This study employed an action research design, in which the instructional intervention was systematically *planned, implemented, observed,* and *reflected* upon in four iterative cycles (Creswell & Creswell, 2018).

3.4.1. Planning

Prior to the instructional intervention, the researchers designed three lesson plans combining problem posing and problem solving methods: "*Understanding Fractions*," "*Proper and Improper Fractions*," and "*Equivalent Fractions: Expansion and Simplification*."

3.4.2. Implementation

The intervention lasted for two weeks, during which three lesson plans were carried out in a sequenced manner across the teaching sessions. Each lesson unit followed the sequence of phases shown in Figure 1, which illustrates how activities were structured to support problem posing and problem solving in a connected and progressive way.

Activating Mathematical Thinking. Each lesson began with a situation or a question presented by the teacher to activate prior knowledge and connect students to the topic. Here, the teacher's role was to stimulate curiosity and ensure that all learners entered the lesson with a shared point of reference.

Guided Problem Work. With teacher support, students worked on visual models and structured problems. The teacher explained, asked guiding questions, and

modelled solution processes, ensuring that students understood the criteria for a solvable problem (clear data, a specific question, and an attainable solution). This direct guidance built a conceptual foundation for later independent and group work.

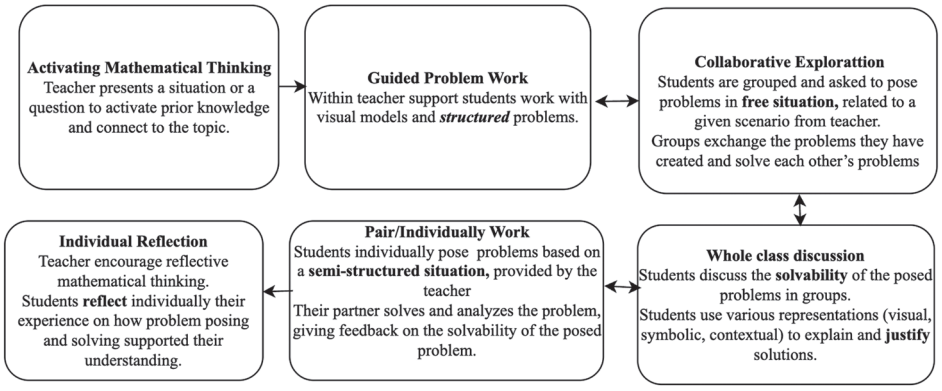


Figure 1. The structure of a lesson plan using problem posing and problem solving teaching methods.

Collaborative Exploration. Students were grouped and asked to pose problems in free situations related to everyday contexts (e.g., food sharing, measuring). Groups exchanged problems and solved each other's tasks. When a problem was unsolvable, students were required to provide written arguments explaining why it could not be solved and what information was needed to resolve it. During this written justification, they were guided to use different mathematical representations, such as visual diagrams, formulas, or number lines, to support their reasoning. The teacher observed group work, intervened when necessary, and encouraged students to refine their problems accordingly. Through these written peer evaluations, combined with teacher feedback, students improved the clarity and solvability of their tasks.

Whole Class Discussion. The teacher facilitated a whole-class dialogue about the solvability of selected problems that students had posed in groups. The discussion was managed in a way that fostered a constructive classroom culture, where students were encouraged to listen to one another, respect different viewpoints, and build upon their peers' contributions. The teacher modelled how to conduct such discussions by highlighting key strategies, clarifying misunderstandings, and ensuring that all students' voices were heard. This collective exchange helped strengthen students' criteria for well-posed problems and fostered a collaborative approach to mathematical reasoning.

Pair/Individual Work. Working in pairs, each student independently posed a problem in a semi-structured situation. These situations required students to pose a problem starting from partial contexts, limited data, or symbolic information. After formulating their problems, they exchanged tasks with their partner, who then solved and analyzed them, providing feedback on the solvability of the tasks. During the activity, the teacher circulated and supported pairs when difficulties

arose, but the responsibility for generating and solving problems rested primarily with the students.

Individual Reflection. At the end of each lesson, the teacher encouraged reflective thinking by asking students to consider how posing problems and solving them had supported their understanding. Students wrote brief reflections and were also assigned homework tasks in which they posed additional problems. The teacher collected these artefacts for record-keeping; however, the data from students' written reflections were not analyzed within the scope of this study.

It is worth noting that throughout all problem posing activities, students were instructed to pose solvable problems, in the sense that they must contain a clear question, sufficient data, and a solution achievable through the mathematical procedures covered in the lesson. If a problem did not meet these criteria, such as a lack of a question, incomplete information, or logical ambiguity, it would be considered unsolvable. Additionally, the posed problems were classified into three types based on the degree of guidance provided during their formulation: structured, free, and semi-structured. A problem was considered *structured* when the student posed the problem based on a known situation, with complete data and a clear question, which could be solved with a known procedure. "*Free*" problems were considered those in which students had complete autonomy in choosing the context, the quantity, and the way of posing the question, without being limited by any given model or structure. *Semi-structured* problems were considered those in which students were given only partial situations, such as a simple and short context, a situation without a clear question, or symbolic data through fractions. Based on this limited information, students were encouraged to formulate a clear mathematical question and create a solvable problem.

3.4.3. Observation

During the intervention, the researchers did not conduct any direct classroom observations. The documentation of classroom processes was based solely on the student artefacts systematically collected by the teacher, including the written tasks completed during classroom activities and homework assignments.

3.4.4. Reflection

Following the intervention, students' written work was reviewed to identify recurring patterns and difficulties in the problems they posed. These findings informed reflection on the instructional process and the difficulties observed during problem posing.

3.5. Data analysis

To answer the research questions, the study used qualitative data. The analysis focused on the mathematical problems students posed during classwork in the

pair/individual work phase. Each posed problem was treated as a separate unit of analysis and examined according to its solvability, context, and structure. In the first stage of analysis, the problems were categorized as solvable or unsolvable. Solvable problems were those that contained sufficient information, a clear mathematical question, and could lead to a valid solution. Unsolvable problems included those with missing information, unclear or incomplete formulation, or mathematical inconsistencies that prevented a valid solution. The solvable problems were then examined in greater detail according to the mathematical ideas and contexts represented in them. Particular attention was given to how fractions were used within the posed problems, including fractions as operators, equivalence of fractions, and fractions as ratios. The problems were also categorized according to their nature as conceptual problems or real-life application problems. Finally, the posed problems were examined according to the structure of the problem posing situation, including free, semi-structured, and structured problems. Frequencies and percentages were used descriptively to show the distribution of problems across the identified categories.

4. Results and interpretations

4.1. Types of posed problems

The analysis of the solvability of the posed problems was initially based on the categorization proposed by Xie and Masingila (2017) who classified posed problems into three categories: (1) solvable mathematics problems, (2) unsolvable mathematics problems (including impossible and insufficient problems), and (3) not a mathematical problem (including non-problems and non-math problems). However, during the coding process, it became evident that the distinction between unsolvable problems and not a mathematical problem was not always clearly observable. As a result, in alignment with how the students' posed problems were interpreted, the analysis proceeded using two broader categories: solvable and unsolvable, as summarized in Table 1.

Table 1. Category of posed problems.

Category	**Solvable**	**Unsolvable**	**Total number of posed problems**
Number of posed problems in each category	48 (73%)	18 (27%)	66

Solvable problems have been considered all those problems that have a clear mathematical structure, contain sufficient information and a defined question, and can be solved based on mathematical knowledge related to the concept of fractions.

Unsolvable problems, in the context of this study, are considered those problems that do not contain a clear question, lack the necessary data for a solution, or are formulated in a way that does not make logical or mathematical sense. Such problems cannot lead to a valid solution, even if their context appears related to

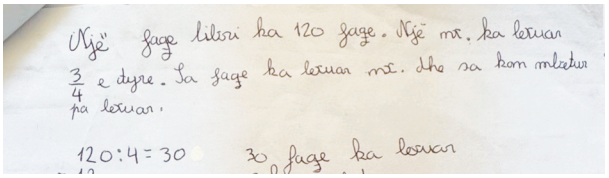
mathematics. This category also encompasses problems that lack mathematical content or structure or do not involve any form of mathematical reasoning.

Analysis of the 66 posed problems showed that the majority were categorized as solvable, with 48 problems (73%) classified as solvable and 18 problems (27%) as unsolvable. These data suggest that most students were generally able to pose problems that were mathematically meaningful and clearly structured, containing sufficient information and a clear mathematical question.

The instructional units implemented during the intervention period oriented the content of the problems formulated by students, particularly in relation to the function of fractions within mathematical contexts. The composition of the posed problems across the two categories (solvable and unsolvable) reflects three distinct uses of fractions: *fractions as operators, equivalence of fractions,* and *fractions as ratios*. An example of each is shown below, to illustrate how students interpreted these different functions of fractions in their posed problems.

Fractions as operators

Example 1

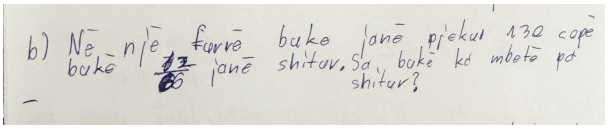


The text in the image is in Albanian, while the English translation is:

"*A book has 120 pages. A student has read 3/4 of it. How many pages has the student read, and how many pages are left to read?*"

This problem exemplifies the use of a fraction as an operator in a familiar context. The student used $3/4$ to determine a part of a known whole and then identified the remaining quantity. The problem is clearly structured, contains sufficient information, and can be solved through a direct application of the fraction to the total number of pages. In this sense, the problem shows that the student was able to use a fraction as an operator within a coherent mathematical situation. However, the task does not require explanation, justification, or comparison of fraction relationships, and therefore provides limited evidence about deeper conceptual understanding. Research on fraction knowledge distinguishes between carrying out procedures and understanding the mathematical relationships underlying them (Hiebert & Lefevre, 1986; Hallett et al., 2010).

Example 2

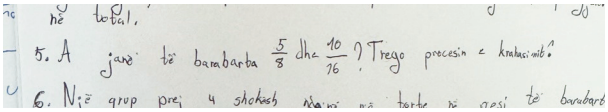


The text in the image is in Albanian, while the English translation is:

"*A bakery baked 130 loaves of bread. 2/6 of them were sold. How many loaves of bread remained unsold?*"

This problem illustrates the use of a fraction as an operator applied to a known whole. The fraction $2/6$ is applied to the total of 130 loaves to determine the number sold, and the remaining quantity is then found from the original total. Unlike a task that asks only for the fractional part, this problem requires an additional step because the quantity that remains must also be determined. The problem therefore shows how a fraction can be used to identify one part of a whole and then relate that result to the remaining quantity.

Equivalence of fractions



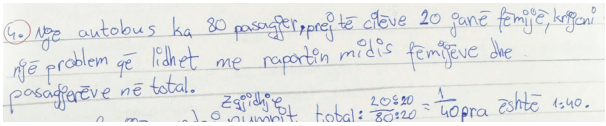
The text in the image is in Albanian, while the English translation is:

"*Are the fractions 5/8 and 10/16 equivalent? Demonstrate the procedure of comparison.*"

This task focuses on fraction equivalence by asking whether $5/8$ and $10/16$ represent the same value. Unlike problems that require only a direct calculation in a familiar context, this problem asks the student to compare two fractions and show the comparison process. This requires attention to the relationship between the numerators and denominators and to the idea that equivalent fractions can represent the same quantity even when they are written differently. The problem therefore provides evidence of how the student used the concept of fraction equivalence when formulating a mathematical task. Its requirement to demonstrate the comparison process also opens the possibility of using numerical procedures or other representations to justify the relationship between the two fractions.

Fractions as ratios

Example 1



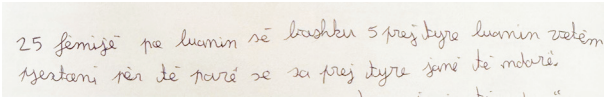
The text in the image is in Albanian, while the English translation is:

"*There are 80 passengers on a bus, 20 of whom are children. Create a problem related to the ratio between the children and the total number of passengers.*"

This task focuses on the use of fractions as ratios by relating the number of children to the total number of passengers. The quantities 20 and 80 form a part-to-whole relationship, which can be represented as $20/80$ and simplified to $1/4$. The task therefore requires attention to the relationship between two quantities rather than only to a direct calculation. It also places the ratio in a familiar context,

showing how fractions can be used to describe the relative size of one group within a larger whole.

Example 2



The text in the image is in Albanian, while the English translation is:

"*Twenty-five children were playing together. Five of them were playing separately. Divide to find what fraction of the children were separated from the group.*"

This problem represents a ratio between a subgroup and the whole group. The five children are considered in relation to the total of 25 children, giving the fraction $5/25$, which can be simplified to $1/5$. Rather than treating the numbers as separate quantities, the problem connects them through a part-to-whole relationship. The context therefore provides a simple way of expressing one group as a fraction of the total group.

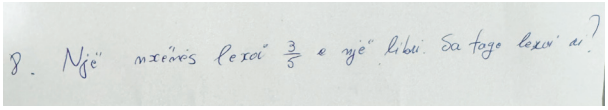
In the following section (*4.1.1*), examples of posed problems are provided for the category of unsolvable problems. However, a deep analysis is done on the category of solvable problems, as they account for the majority of the data. The solvable problems were classified along two dimensions. First, they were categorized, based on the nature of cognitive demand, into two groups: *conceptual problems* (those involving questions such as *why*, *explain*, or *justify*) and *real-life application problems* (those involving questions such as *how much*, *how many*, or *what is left*). Second, the solvable problems were also classified according to their structural form as *free*, *semi-structured* or *structured problems*, depending on the level of guidance provided during their formulation. A more detailed analysis of these categories is presented in *Sections 4.2* and *4.3*.

4.1.1. Examples of unsolvable problems

Of the 18 unsolvable problems, seven lacked sufficient data or had unclear questions, rendering them incomplete and unsolvable (see below category I). The remaining eleven problems (see below category II) were formulated in an unclear or illogical way, including mathematical inconsistencies that contradicted the relationships represented by the fractions.

Although these problems were classified as unsolvable, they provide valuable insights into how students understand and formulate mathematical content. Their errors help identify gaps in thinking, misunderstandings that may have as well as the presentation of real situations in a mathematical context. In this regard, unsolvable problems should not be seen as failures, but as opportunities to reflect and learn, thereby emphasizing the importance of learning from mistakes.

Category I: Lack of Sufficient Data



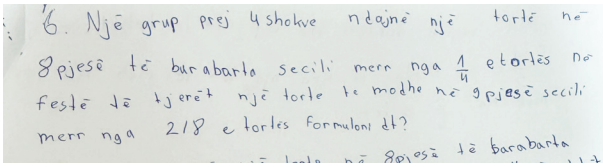
The text in the image is in Albanian, while the English translation is:

"*A student read 3/5 of a book. How many pages did the student read?*"

This problem was categorized as unsolvable because it does not provide sufficient information to determine a unique answer. Although the fraction $3/5$ is given, the total number of pages in the book is not specified. Without knowing the whole to which the fraction refers, the number of pages read cannot be calculated. Although others may perceive this problem as open-ended, as it allows students to make assumptions or explore different cases, in the context of this study, it is categorized as unsolvable. Xie and Masingila (2017) similarly distinguish insufficient problems as mathematical problems that cannot be solved because the information provided is not sufficient. In this case, the difficulty lies not in the fraction itself, but in the missing information needed to connect the fraction to a specific whole. Missing all the data in the task is one of the most common errors that occur during problem posing (Mallart et al., 2018). This may arise from the failure to recognize the necessary components of a mathematical problem, concentrating on replicating problems similar to those learned in class, but lacking a full understanding of the fundamental criteria for a valid solution.

Category II: Lack of Mathematical Consistency

Example 1



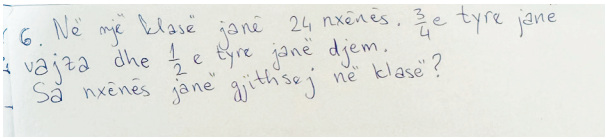
The text in the image is in Albanian, while the English translation is:

"*A group of four friends divide a cake into eight equal parts, and each receives 1/4 of the cake. At another party, a larger cake is divided into nine equal parts, and each receives 2/8 of the cake.*"

This problem was categorized as unsolvable because it does not contain a clear mathematical question and includes an unclear mathematical relationship in the way the fractions are represented. Although the problem presents a recognizable context and numerical information, it does not specify what needs to be determined. In addition, the second part describes a cake divided into nine equal parts while each share is expressed as $2/8$ of the cake, making the relationship between the stated partition and the fraction unclear. Previous research on problem posing has emphasized clarity and mathematical accuracy as important criteria for evaluating

posed problems (Özçakır Sümen, 2021). Xie and Masingila (2017) also identified difficulties in fraction problem posing related to the meaning of fractions, mathematical operations, and the unit to which a fraction refers. An unclear question and an unclear fraction relationship prevent the problem from leading to the correct solution.

Example 2



The text in the image is in Albanian, while the English translation is:

"*There are 24 students in a class. 3/4 of them are girls and 1/2 of them are boys. How many students are there altogether?*"

This problem was categorized as unsolvable because the fractional relationships given in the problem are mathematically inconsistent. If $3/4$ of the 24 students are girls and $1/2$ are boys, the two groups would represent $5/4$ of the whole class, which exceeds the total number of students. Therefore, the fractions cannot simultaneously describe two non-overlapping groups within the same class. In addition, the question asks for the total number of students even though this information is already provided at the beginning of the problem. The difficulty is therefore not a lack of numerical information, but an inconsistency in coordinating the fractions with the whole they are intended to represent. Xie and Masingila (2017) similarly found that difficulties in fraction problem posing can reveal misunderstandings in how fractions are related to quantities and to the unit or whole being considered. In this sense, the problem illustrates how an apparently familiar context can still lead to an unsolvable formulation when the fractional relationships are not mathematically coherent.

The comparison between the solvable and unsolvable examples shows that the main difference was not simply the presence of fractions or a familiar context, but whether the information and mathematical relationships in the posed problem led to a clear solution. Solvable problems included sufficient information, a clear question, and fraction relationships that were mathematically consistent. In contrast, the unsolvable problems contained difficulties such as missing information, unclear questions or formulation, and mathematical inconsistencies. Some unsolvable problems still included familiar contexts and appropriate fraction notation, but the information provided did not form a problem that could lead to a valid solution. This distinction shows that using fractions within a problem was not sufficient on its own; the information, question, and mathematical relationships also needed to be appropriately connected.

4.2. The context of solvable posed problems: conceptual and real-life applications

After analyzing the "solvability" of the posed problems, we decided to focus the analysis also on the context and type of mathematical knowledge they elicited. The problems formulated by students were examined to see whether they had developed a conceptual understanding of fractions or were primarily situated within real-life applications. Problems based on conceptual knowledge (hereinafter referred to as *conceptual problems*) in mathematics are distinguished by the nature of knowledge required and are referred to as an understanding of concepts, principles, and relationships within a specific domain. These problems encourage students to develop relational understanding and to explain, justify, or reason about a mathematical idea beyond simply performing calculations (Hiebert & Lefevre, 1986). Real-life application problems present mathematical concepts in practical, everyday situations, making mathematics meaningful and relevant. The effectiveness of a context in making a problem seem real is influenced by how students interact with it. This interaction is often individual and can lead to unexpected interpretations of the task (Boaler, 1993).

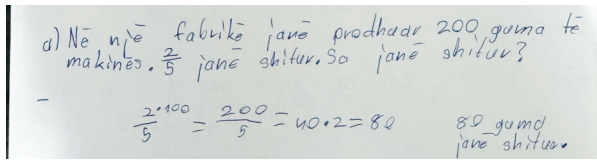
Table 2. Comparison of conceptual problems and real-life application problems. *Source (Boaler, 1993; Rittle-Johnson et al., 2001).*

Feature	Conceptual Problems	Real-Life Applications
Definition	Problems that require reasoning, explanation, and mathematical justifications	Problems integrated in practical, everyday contexts where concepts are applied to real-life situations
Focus	Explaining mathematical ideas	Applying mathematical concepts to practical contexts
Cognitive Demand	Higher-Requires abstract thinking and deep understanding of mathematical concepts	Moderate to High-Requires applying fraction knowledge in meaningful contexts
Question Type	"Why?" "Explain." "Justify."	"How much?" "How many?" "What is left?"
Examples	Explain why $5/7$ is greater than $3/5$ using a visual model; explain why multiplying by $2/3$ makes a number smaller.	Dividing a pizza, scaling a recipe, splitting money. A farmer divides his land into 5 equal parts. If he plants wheat on $3/5$ of the land, how much land is left for vegetables?
Learning Outcome	Develops analytical reasoning, conceptual mastery of fractions and strengthens problem solving abilities that require explanation and justification	Enhances problem solving skills and connects mathematical concepts to real-life situations

Table 2 breaks down the distinctions between conceptual problems and real-life application problems. The theoretical part of the table is based on Rittle-Johnson et al. (2001) and Boaler (1993). Rittle-Johnson et al. (2001) analyze the differences between conceptual and procedural knowledge, showing that conceptual problems help deepen mathematical understanding and develop analytical reasoning, while real-life application problems help in the practical application of mathematical knowledge in real-life contexts. Boaler (1993) examines the use of mathematical

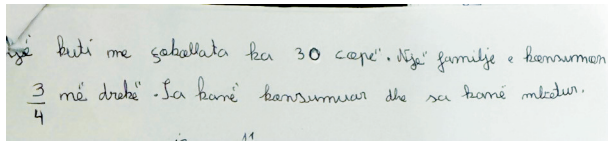
problems presented in real-life contexts and shows that the value of such contexts depends on how students interpret and relate to the situations in which the mathematics is embedded.

The categories shown in Table 2 are described below, followed by examples illustrating how the posed problems were classified. Such examples include:



The text in the image is in Albanian, while the English translation is:

"*In a factory, 200 machines were produced. Two-fifths of them were sold. How many were sold?*"

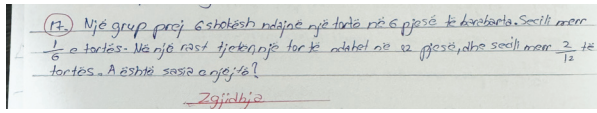


The text in the image is in Albanian, while the English translation is:

"*A box of chocolates contains 30 pieces. A family consumes 3/4 of them. How many chocolates have been consumed, and how many are left?*"

These examples illustrate how fractions were embedded in familiar situations involving quantities, such as production and consumption. In both problems, the fraction functions as an operator applied to a known whole, and the required solution involves determining a part of that whole. Although the contexts differ, the mathematical structure is similar: a total quantity is given, a fraction of that quantity is identified, and the solution requires calculating the corresponding part. In the second example, an additional step is required because the remaining quantity must also be determined. The problems are clearly formulated and rely mainly on direct calculations rather than on explaining or justifying relationships between fractions. This suggests that, within the real-life application category, familiar contexts were often used to frame procedures that students already knew how to apply. At the same time, the two examples show that real-life application problems can vary slightly in the number of steps required, even when they are based on a similar mathematical relationship. This pattern is consistent with the predominance of real-life application problems identified in the data and shows that everyday situations provided a common setting for posing fraction problems. Silver and Cai (1996) similarly noted that students' posed problems are often influenced by their prior experiences with familiar problem structures.

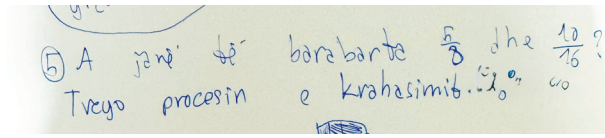
On the other hand, conceptual problems identified in students' written work included questions that required analysis and reasoning about relationships between fractions, such as:



The text in the image is in Albanian, while the English translation is:

"*A group of six friends divides a cake into six equal parts, and each receives 1/6 of the cake. In another case, a larger cake is divided into 12 equal parts, and each receives 2/12 of the cake. Is the amount the same?*"

This problem was categorized as conceptual because, although it is presented in a familiar real-life context, its main mathematical demand is to compare two fraction representations and determine whether they represent the same quantity. The problem therefore focuses on the relationship between $1/6$ and $2/12$, rather than on the direct application of a procedure to find a quantity. Recognizing that these fractions are equivalent requires attention to the relationship between the numerator and denominator and to the idea that the same quantity can be represented by different fractions.



The text in the image is in Albanian, while the English translation is:

"*Are the fractions 5/8 and 10/16 equivalent? Show the comparison process.*"

This problem was categorized as conceptual because it focuses on fraction equivalence and asks whether two fractions represent the same value. It requires students to examine the relationship between the fractions rather than simply carry out a calculation, which is consistent with conceptual knowledge as understanding mathematical relationships (Hiebert & Lefevre, 1986).

Such problems require students to engage in higher-level thinking, analyzing the basic concepts of fractions and the relationships between them. However, the small number of such problems indicates that students tend to avoid formulating questions that require in-depth and abstract reasoning. Similarly, Crespo (2003) implies that students often feel more confident in creating problems based on familiar contexts, avoiding conceptual challenges that require a deeper understanding of mathematics.

The comparison between these examples from both categories shows that the distinction between conceptual and real-life application problems was based mainly on the mathematical demand of the problem rather than simply on whether a familiar context was present. In the conceptual example, the central focus is on comparing two fractions and examining whether they represent the same value. The mathematical relationship itself is therefore the main part of the task. In the real-life application example, the fraction is used within a familiar situation to determine a quantity from a known whole. The main demand is to apply the fraction in order

to obtain a numerical result. This distinction is important because a contextual problem may still involve a conceptual relationship, while a real-life application problem may mainly require the direct use of a known procedure. Table 3 shows how the solvable problems were distributed across these two categories.

Table 3. Distribution of solvable problems based on their context.

Category	Conceptual Problems	Real-Life Applications	Total number of solvable problems
Solvable problems	7 (15%)	41 (85%)	48

The higher number of real-life application problems shows that most of the posed problems were formulated in everyday contexts, such as cooking, shopping, or sharing. However, the smaller number of conceptual problems indicates that tasks requiring comparison, explanation, or reasoning about fraction relationships were less common. These findings suggest that students should be encouraged more often to formulate conceptual problems that require reasoning and attention to mathematical relationships. Lavy and Bershadsky (2003) stress the value of problem posing in teaching and learning as a way of supporting a more comprehensive understanding of mathematical ideas. Teachers can accomplish this by promoting activities that require students to create and solve conceptual problems, pushing them to think beyond common applications; encouraging open discussions about the fundamental ideas of fractions and their relationships; offering support and guidance to students as they work through problems; assisting them in developing the skills necessary to formulate questions that call for in-depth reasoning.

4.3. The structure of solvable problems

The structure of the problems formulated by students significantly reflects their understanding of mathematical concepts and how their learning experiences influence the shape of making meaning.

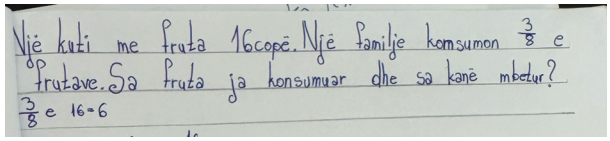
According to Stoyanova and Ellerton (1996), structured problems are those in which the student creates a problem with a complete structure: complete data, a clear requirement, and a known situation. Semi-structured problems involve scenarios where students pose problems based on a given figure, table, or equation. Meanwhile, free problems are those in which students are asked to pose problems from a given situation, with some instruction possibly provided to prompt a specific activity. In this case, students have a high degree of openness in creating content, given the freedom they have to independently pose a problem.

Although students were engaged in posing problems in all three planned situations, *free, semi-structured,* and *structured,* during the intervention phase, the analysis of the collected data revealed that the posed problems were distributed into only two categories: *structured* and *free*. No problems were identified that met the criteria to be classified as semi-structured. This may be a result of the specific nature of semi-structured situations, in which students are instructed to

pose problems based on a situation provided by the teacher (such as a description, a report, or a result). A possible explanation for this lack may be related to how students responded to the partial situations presented by the teacher. Instead of posing a "prompt" or question that would enable the solution of a problem, in many cases, they limited themselves to reproducing the given information without further developing it into a full mathematical problem. This suggests that, at this stage of the learning process, students may have needed more support or clarification about what constitutes a problem formulated in a semi-structured situation.

To illustrate how the problems posed by students were classified, an example is presented for each category of two problem structures.

An example of *structured* problem:

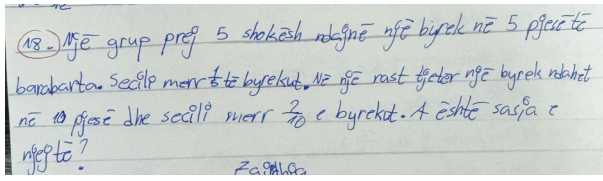


The text in the image is in Albanian, while the English translation is:

"*A box contains 16 pieces of fruit. A family consumes 3/8 of the fruit. How many pieces of fruit have been consumed, and how many are left?*"

This problem was classified as structured because it follows a familiar problem form in which a fraction is applied to a known whole. The total quantity is given, the fraction $3/8$ indicates the part that is consumed, and the questions ask for the consumed and remaining quantities. The mathematical relationship is therefore clearly defined and leads to a direct solution. Although the context relates to everyday life, the problem structure remains close to familiar classroom exercises, where the necessary information and required calculations are clearly specified.

An example of *free* problem:



The text in the image is in Albanian, while the English translation is:

"*A group of five friends divides a burek into five equal parts. Each person receives 1/5 of the burek. In another case, a burek is divided into 10 parts, and each person receives 2/10 of the burek. Is the amount the same?*"

This problem was classified as free because the complete mathematical situation was formulated without a predefined problem structure. Notably, the two given fractions, $1/5$ and $2/10$, were incorporated into a meaningful everyday context involving the sharing of a burek. The problem then asks whether the two fractional quantities represent the same amount, linking the contextual situation

with the mathematical idea of equivalent fractions. In contrast to the structured example, the mathematical relationship is not presented as a direct calculation based on a given whole but is embedded in a situation created around the two fractions. This shows how the given fractions were used to construct a complete contextual problem rather than being treated only as isolated numerical expressions.

The contrast between these two examples also shows that the distinction between structured and free problems was not based simply on whether a problem had a real-life context. Both examples were formulated in familiar contexts. The difference lies in how much of the mathematical situation had to be constructed. In the structured example, the quantities, fraction relationship, and required calculations follow a familiar form. In the free example, the two fractions were incorporated into a context and connected through a question about whether they represent the same amount. Thus, the classification was based primarily on the way the problem was constructed rather than on the presence or absence of a context. Table 4 presents the overall distribution of solvable problems across these two structural categories.

Table 4. Distribution of solvable problems based on their structure.

Category	Structured	Free	Total
Number of Solvable Problems	39	9	48
Percentage (%)	81%	19%	100%

As shown in the table, the majority of problems (81%) fall in the structured category, indicating that most students tend to pose problems that mirror the most commonly used forms of exercises found in textbooks. As Ellerton (2013) points out, many students have been "socialized" to believe that mathematical problems are always created by others and only need to be solved. This "culture of passivity" of mathematics helps to explain why most students in this study formulated structured problems, avoiding those that require greater initiative or deeper abstraction. In this sense, structured problems not only reflect the level of thinking but also illustrate how mathematics learning is experienced by students, often as an activity of solving, rather than as a shape of meaning.

Of the 48 solvable problems, 9 (19%) were classified as free problems, meaning that students created problems without relying on any existing model or structure. This result indicates that a smaller proportion of students were willing to explore beyond the tasks taught, taking care of all aspects of the problems they created. This suggests that the problem posing and problem solving methods used during the intervention facilitated students in forming and posing mathematical content of problems themselves. This category is particularly important because it reflects students' ability to independently generate mathematical scenarios in which known concepts are applied in varied problem situations. It suggests that the student not only understands the underlying mathematical ideas but is also capable of reformulating them into novel problem contexts that require the integration of learned knowledge. According to the literature (Ellerton, 2013; Stoyanova & Ellerton, 1996), posed problems in free situations are a high indicator of conceptual reasoning and mathematical imagination, putting students in the role of creators of

scenarios that can be related to real life or personal contexts. However, the percentage (19%) is relatively low compared to structured problems, indicating that students feel more comfortable relying on familiar contexts when posing problems. This may reflect the need for more pedagogical support in helping students develop open-ended problem posing abilities, as well as limitations that may arise from previous learning experiences that favor posing closed-ended problem structures.

The high number of structured problems in this study suggests that students are still at the stage where they need clear instructions to pose mathematical problems that encourage deeper mathematical thinking. From a didactic perspective, the results support the need to establish a balance between providing structure and encouraging independent exploration. The consistent integration of problem posing and problem solving methods into teaching can facilitate the progressive development of students' abilities to pose problems with varying degrees of openness.

4.4. Answers to the research questions

The findings provide an overview of the types of fraction problems posed by sixth-grade students during the intervention. In relation to the first research question, most of the collected problems were solvable, while a smaller number were classified as unsolvable. The solvable problems generally included the information and mathematical relationships needed to reach a solution. They covered different uses of fractions, including fractions as operators, equivalent fractions, and fractions used to express relationships between quantities. This variation shows that the posed problems were not limited to a single form of fraction task, although some forms appeared more frequently than others.

The unsolvable problems provide a different perspective on the problems posed during the intervention. These problems included missing information, unclear questions or formulation, and mathematical inconsistencies. In some cases, the context and fraction notation were recognizable, but the information given was not sufficient or appropriately related to produce a valid solution. In other cases, the difficulty appeared in the way the quantities or fractional relationships were expressed. Therefore, the distinction between solvable and unsolvable problems was not based simply on whether fractions were included, but on whether the information and mathematical relationships formed a problem that could be solved.

In relation to the second research question, the solvable problems showed clear differences in their nature. Real-life application problems represented the larger category, with 41 of the 48 solvable problems, while seven were classified as conceptual. The real-life application problems commonly placed fractions in familiar contexts involving quantities, sharing, consumption, or other everyday situations. Fractions were frequently used as operators applied to a known whole. By contrast, the conceptual problems focused more directly on mathematical relationships and included tasks involving comparison, equivalence, reasoning, or explanation. The smaller number of conceptual problems indicates that this type of problem was less common in the collected data.

An important point emerging from this distinction is that the presence of an everyday context did not by itself determine whether a problem was conceptual or a real-life application problem. The classification depended mainly on the mathematical demand of the problem. Some contextual problems could still direct attention to relationships between fractions, whereas many real-life application problems mainly required the use of a known procedure to determine a quantity. This distinction helped to describe more clearly the nature of the problems rather than classifying them only according to the context in which they were presented.

The second research question also concerned the structure of the posed problems. Most of the solvable problems were classified as structured, with 39 problems in this category, while nine were classified as free. No collected problems met the criteria for the semi-structured category, although students had been engaged in all three types of problem posing situations during the intervention. The predominance of structured problems shows that familiar problem forms remained common in the problems that were posed. The free problems, although fewer in number, differed in that the mathematical situation and question were formulated with less dependence on a familiar predefined form.

The absence of semi-structured problems is also relevant to the second research question. Although semi-structured situations were included during the intervention, the collected responses did not develop into problems that met the criteria for this category. In some cases, partial information provided as a starting point was reproduced rather than developed into a complete mathematical problem. This suggests that the distinction between participating in a semi-structured problem posing situation and actually producing a problem that can be classified as semi-structured is important when interpreting the findings.

The findings answer the two research questions by showing both the kinds of fraction problems that were posed and the differences in their nature and structure. The collected problems were characterized mainly by solvable, real-life application, and structured problems, while conceptual and free problems were less frequent and no semi-structured problems were identified. At the same time, the unsolvable problems showed several difficulties that appeared during problem formulation. The analysis therefore provides information not only about the categories into which the posed problems fell, but also about the different ways fractions were represented and the difficulties that became visible when mathematical problems were posed.

5. Discussion

This study explored the ability of sixth-grade students to pose fraction problems while engaging in problem posing and problem solving activities. The qualitative analysis of students' posed problems provides insight into the types of problems students formulated and the difficulties that emerged during problem posing. The majority of the students' posed problems were solvable; however, most were real-life application problems, suggesting that students may need additional guidance to formulate conceptual problems.

The posed problems provide examples of how students used fraction concepts when posing their own mathematical problems. In particular, the problems involved fractions as operators, ratios, and equivalent forms, reflecting several of the meanings associated with fraction knowledge (Manfreda Kolar et al., 2018). These examples show that students were able to use fractions in different mathematical roles rather than within a single type of situation. However, the presence of a particular fraction meaning in a posed problem should not be interpreted as evidence of complete conceptual understanding. Rather, it shows how students were able to use specific fraction relationships within the problems they created. This distinction is important because conceptual and procedural knowledge are related but do not necessarily develop in the same way (Hallett et al., 2010; Lenz & Wittmann, 2020).

From this perspective, the predominance of real-life application problems suggests that students most often expressed their fraction knowledge through familiar contextual situations. These problems generally involved the direct use of a fraction within a quantity, such as finding a part of a whole or determining what remained. The use of familiar contexts can make mathematical situations more accessible, although the mathematical value of such contexts depends on how students interpret and work with them (Boaler, 1993). By contrast, the smaller number of conceptual problems required students to compare, explain, or justify relationships between fractions. Such demands are more closely related to conceptual knowledge, which involves understanding mathematical relationships rather than only carrying out procedures (Hiebert & Lefevre, 1986). Although these problems were less frequent, they are important because they show instances in which students engaged more directly with relationships between fractions. The contrast between these two types of posed problems therefore provides a more differentiated view of how fraction understanding was expressed in students' work.

Posing problems that demand justification is an advanced ability that requires not only an understanding of mathematical concepts but also the capacity to articulate them in a manner that provokes critical thinking in others. The limited number of conceptual problems in this study may reflect students' more limited experience in posing problems that require mathematical explanation and reasoning. The conditions under which problem posing takes place may influence the kinds of problems students create. In addition to the nature of the problems, this study also examined their structure. The results showed that the majority of the problems posed by sixth-grade students were structured, while the number of free problems was smaller. This distribution highlights that students tend to pose problems based on known concepts introduced during classroom lectures. The small number of free problems, which require the formulation of the situation and the question itself, highlights the need for teaching strategies that encourage greater autonomy and flexibility in students' mathematical thinking. The predominance of structured problems can also be interpreted in relation to the conditions under which problem posing takes place. Silber and Cai (2017) found that different problem posing situations can influence how closely participants attend to the mathematical relationships involved in a task. Similarly, Martínez and Blanco (2021), working specifically with fraction problems, examined how different forms of support shaped the kinds of problems students generated. In the present study, the larger

number of structured problems suggests that students relied more often on familiar formats and clearly defined conditions when posing problems. This does not necessarily reduce the value of structured problem posing; rather, structured situations may provide an accessible starting point for students with limited experience creating mathematical problems independently. By contrast, the smaller number of free problems indicates that posing a problem with less external guidance requires students to make more decisions about the context, numerical information, mathematical relationships, and the question itself.

When students are given opportunities to work with less structured prompts and questions that require reasoning, they may be encouraged to move beyond direct applications and familiar problem formats. The type of prompt used by the teacher is important because it can influence the kinds of problems students generate and the mathematical relationships they attend to (Silber & Cai, 2017; Cai & Hwang, 2023). This suggests that carefully varying the level of guidance and the demands of the prompt may help students formulate a broader range of problems, including those that require explanation, justification, and more independent mathematical thinking.

The unsolvable problems provide an additional perspective on the difficulties students experienced when formulating fraction problems. Although these problems represented a smaller part of the data, they revealed difficulties related to missing information, unclear questions, and mathematical inconsistency. Such difficulties show that posing a problem requires more than inserting numbers or fractions into a familiar context; students must coordinate the information, mathematical relationships, and the question so that the problem becomes meaningful and solvable. Similar dimensions, including clarity, mathematical accuracy, complexity, and consistency with the given conditions, were considered by Özçakır Sümen (2021) in analyzing students' fraction problem posing. In the present study, the unsolvable problems therefore provide useful evidence about where difficulties arose in the process of posing solvable mathematical problems. These difficulties also contribute to a more nuanced view of students' understanding of fractions. In some unsolvable problems, students used fraction notation and familiar contexts, but the relationships between the quantities were not always coordinated in a mathematically consistent way. This is important because understanding fractions involves recognizing and coordinating different mathematical relationships, rather than simply applying fraction notation or procedures (Hiebert & Lefevre, 1986; Manfreda Kolar et al., 2018). Xie and Masingila (2017) similarly showed that difficulties emerging during fraction problem posing can reveal misunderstandings related to fraction meanings, operations, and the unit to which a fraction refers. In this sense, posed problems can make visible both what students are able to express mathematically and where difficulties in their understanding become apparent.

The findings of this study also underscore the pedagogical value of integrating problem posing into mathematics instruction and provide guidance for designing lesson plans that bring together problem posing and problem solving methods. Viewing the intervention as an interconnected cycle of phases from teacher-led structured problem solving to collaborative/pair/individual formulation of free

and semi-structured problems to individual reflection, shows that combining problem posing and problem solving methods exposed students to different forms of mathematical activity. However, the problems collected during the intervention were predominantly structured, while free problems were less frequent and no problems were classified as semi-structured. Future work should also refine the integrated problem posing and solving methods to cultivate a wider spectrum of posing abilities. Some improvements can be particularly notable, such as establishing clear criteria that distinguish between "conceptual problems" and "real-world problems". The results showed that students primarily formulated tasks based on real-life applications, while problems that required conceptual explanations were few. To balance this tendency, the teacher can provide groups with a list of simplified criteria that distinguish between these two categories (e.g., reasoning requirement, variety of strategies, type of question) printed on a guide sheet that remains on the table throughout the activity. This visual "map" helps students: Self-assess whether their problem is still a real life application or reaches a conceptual level; Review problems in real time within the group without waiting for feedback from the teacher; Unify the understanding of the characteristics that constitute a conceptual problem, maintaining coherence between student groups and throughout the lessons. To guide students toward the formulation of semi-structured and free-form problems, the teacher could present "semi-ready" problems. Initially, they are given the story and the data, while the question is left blank for the groups to write themselves. In the next task, only the question is provided and the students choose the data themselves. Next, the teacher simply provides a context sentence or a picture, while the groups draft all the remaining parts of the problem. After each step, they exchange work with another group, receive suggestions to make the task even less guided, and revise it. This gradual fading of support, combined with peer feedback, moves students away from imitating familiar examples and leads them step by step toward the independent formulation of semi-structured and free problems.

6. Limitations and recommendations of the study

To place the findings of this study in their proper context, several limitations should be acknowledged. First, the selection of participants was purposeful with a single class of sixth-grade students, which limits the possibility of generalizing the results to a larger group of students. Second, the qualitative analysis of the nature and structure of the posed problems was based only on the students' written work. The use of additional data sources, such as classroom observations, student interviews, or metacognitive strategies, could help in a deeper understanding of how students thought when formulating the problems. The use of metacognitive strategies would enable teachers to gain insight into how students think during problem posing, thereby providing them with the opportunity to offer the necessary assistance when students apply mathematical concepts inappropriately. Additionally, metacognitive strategies would help students develop the ability to self-regulate their problem posing and exercise self-control during this process. Future studies could examine how the use of simple metacognitive guides during problem posing

affects the quality of the problems created, students' conceptual understanding, and the development of their mathematical autonomy.

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Contact addresses:

Blerina Tafolli Avdylaj
Faculty of Education, University of Prishtina "Hasan Prishtina"
Agim Ramadani, 10 000 Prishtina, Kosovo
e-mail: blerina.tafolli@uni-pr.edu

Tatjana Hodnik
Faculty of Education, University of Ljubljana
Kardeljeva ploščad 16, 1000 Ljubljana, Slovenia
e-mail: tatjana.hodnik@pef.uni-lj.si

Të kuptuarit e nxënësve për thyesat përmes shtrimit dhe zgjidhjes së problemeve

Blerina Tafolli Avdylaj[1] dhe Tatjana Hodnik[2]

[1]Fakulteti i Edukimit, Universiteti i Prishtinës "Hasan Prishtina", Kosova
[2]Fakulteti i Edukimit, Universiteti i Ljubljanës, Slovenia

Abstrakti. Zhvillimi i kompetencës së nxënësve për të shtruar dhe zgjidhur probleme matematikore përbën një aspekt të rëndësishëm në procesin e të nxënit, pasi ajo mund të mbështesë të kuptuarit më të thellë të koncepteve matematikore. Prandaj, është e rëndësishme që nxënësit të ekspozohen vazhdimisht në aktiviteteve që nxisin aftësinë për të shtruar dhe zgjidhur probleme. Këto aktivitete nuk duhet të trajtohen thjesht si ushtrime shtesë apo si detyra përmbyllëse në fund të orëve mësimore, por duhet të integrohen në mënyrë sistematike dhe kuptimplotë në të gjitha fazat e procesit mësimor.

Ky studim shqyrton efektin e përdorimit të metodës së shtrimit dhe zgjidhjes së problemeve në mësimdhënien e thyesave, dhe ndikimin e kësaj metode në zhvillimin e kompetencës për shtrimin e problemeve te nxënësit e klasës së gjashtë. Për më tepër, hulumtimi analizon strukturën dhe natyrën e problemeve të formuluara nga vetë nxënësit gjatë fazës së intervenimit. Nëpërmjet një analize cilësore të 66 problemeve të formuluara (nga njëzet nxënës), ato janë klasifikuar si të zgjidhshme, të pazgjidhshme ose jo-matematikore. Nga 48 problemet e zgjidhshme, shumica (85%) përfaqësojnë probleme që lidhen me aplikime në jetën e përditshme, ku thyesat përdoren si operatorë në kontekste të njohura. Ndërsa, vetëm 15% e problemeve kishin natyrë konceptuale dhe kërkonin arsyetim, krahasim ose shpjegim të marrëdhënieve ndërmjet thyesave. Këto rezultate sugjerojnë se, megjithëse nxënësit mund të formulojnë probleme matematikisht të vlefshme, problemet e tyre shpesh vendosen në kontekste të jetës së përditshme dhe zgjidhen përmes një numri të vogël hapash proceduralë, në vend që të nxisin një eksplorim më të thellë konceptual të ideve matematikore.

Të gjitha problemet e zgjidhshme u klasifikuan më tej në dy grupe: *probleme të strukturuara (81%) dhe probleme të lira (19%)*. Numri i vogël i problemeve të pazgjidhshme (27%) sugjeron se nxënësit janë në gjendje të formulojnë probleme të zgjidhshme, por gjithashtu thekson nevojën për të përmirësuar aftësitë e tyre në strukturimin e problemeve, në mënyrë që të sigurohet përputhja me konceptet matematikore të përdorura. Këto gjetje theksojnë nevojën për të dizajnuar procesin e mësimdhënies dhe të të nxënit në mënyrë që nxënësit të ekspozohen ndaj modeleve konceptuale të shtrimit dhe zgjid-

hjes së problemeve, të cilat mbështesin përdorimin e përfaqësimeve të ndryshme të koncepteve matematikore, si dhe problemeve që nxisin arsyetimin.

Një përfundim tjetër i këtij studimi është nevoja për t'u ofruar mësimdhënësve mbështetje didaktike dhe udhëzime të qarta për të nxitur një qasje më të thellë dhe matematikisht reflektuese ndaj shtrimit të problemeve në klasë. Vlen të theksohet, dhe kjo është vërejtur edhe në gjetjet e këtij studimi, se nxënësit nganjëherë mund të jenë të paqartë lidhur me pritjet e mësimdhënësit; ata mund të mos e dinë se çfarë pritet prej tyre nëse kjo nuk bëhet e qartë nga mësimdhënësi. Në rrethana të tjera, megjithatë, problemet e lidhura me jetën reale mund të jenë të dëshirueshme dhe nxënësit mund të vlerësohen për formulimin e tyre.

Fjalët kyçe: thyesat, shtrimi i problemeve, të kuptuarit konceptual, aplikimet në jetën e përditshme, problemet e lira, gjysmë të strukturuara dhe të strukturuara

The Impact of Contextual Tasks on Students' Attitudes Toward Learning Mathematics

Mihaela Bosak and Marija Jakuš*

Faculty of Organization and Informatics, University of Zagreb, Croatia

Abstract. It is well known that contextual tasks with real-life problems can enhance engagement and improve academic achievement. However, there is still debate about the extent to which such an approach influences students' long-term attitudes toward mathematics. This study examines the impact of contextual mathematical tasks on students' attitudes toward learning mathematics, with a particular focus on their perception of the usefulness of mathematics, motivation, and learning satisfaction. The aim of the study was to explore the relationships between contextual tasks, perceived usefulness, motivation, satisfaction, and overall attitude toward learning mathematics.

An empirical study was conducted among first-year professional study students at the Faculty of Organization and Informatics, University of Zagreb, using a survey questionnaire. The collected data were analyzed using the structural modeling method with the SmartPLS4 software. The research model included four constructs: perceived usefulness, motivation, satisfaction, and overall attitude toward learning mathematics. In addition to the constructs, two manifest variables were observed: satisfaction with contextual tasks and the number of solved tasks.

The results of the analysis showed that perceived usefulness significantly influences students' motivation, and a high level of motivation leads to greater satisfaction with learning mathematics. Furthermore, satisfaction with learning mathematical content is positively associated with students' overall attitude toward mathematics. Data analysis also confirmed a significant relationship between satisfaction with contextual tasks and the perceived usefulness of mathematics, suggesting that students who solve contextual tasks with real-life problems are more likely to perceive mathematics as useful. However, the research results did not show a significant direct relationship between the number of solved contextual tasks and the perceived usefulness of mathematics, nor between satisfaction with contextual tasks and students' overall attitude toward mathematics.

*Corresponding author.

These research findings highlight the importance of a contextual approach in mathematics teaching, as it can enhance the perception of the usefulness of mathematics, thereby increasing students' motivation and satisfaction and indirectly contributing to a more positive attitude toward mathematics.

Keywords: mathematics, contextual tasks, attitudes toward learning, perceived usefulness, motivation

1. Introduction

The use of contextual tasks in mathematics education has gained significant attention for its potential to enhance students' engagement, motivation, and overall attitudes toward learning mathematics. Attitudes toward learning mathematics through real-world problem-based tasks are generally positive, as such tasks increase students' engagement, motivation, and perceived value of mathematics by connecting mathematical concepts to their everyday experiences. This approach helps students recognize the relevance of mathematics to their lives, which can lead to more positive attitudes toward the subject. Students generally express positive attitudes toward learning mathematics through such tasks. These tasks are often experienced as challenging but also rewarding, contributing to greater self-confidence and competence in mathematics. However, while real-world tasks have been shown to positively influence motivation and engagement, research indicates that their effectiveness can vary depending on context and implementation. For example, the role of contextual tasks in fostering a more positive evaluation of mathematics among students remains a matter of debate, as some research shows that these tasks do not always lead to increased motivation. In addition, the success of this approach may depend on factors such as the educational objectives of the assignment and how teachers integrate them into the context of the curriculum.

2. Literature review

Research shows that contextual tasks, such as problem solving and applying mathematics to real-world situations, lead to greater student engagement and motivation. For example, Australian primary school students demonstrated positive attitudes toward learning through problem-solving tasks, with younger students showing greater enthusiasm (Russo & Minas, 2021). Similarly, UK vocational school students showed improved attitudes when mathematical topics were presented in real-world contexts (Dalby, 2015). Here, we define "attitudes toward learning mathematics" as affective responses that include positive or negative feelings of moderate intensity and reasonable stability (McLeod, 1992). In the context of mathematics education, McLeod positions attitudes between beliefs, which are cognitively stable, and emotions, which are intense but short-lived, emphasizing their role in mathematics learning. Research also shows that students who engage

in contextual tasks in class often achieve better results in mathematics. A study conducted in Uganda found a direct positive association between students' attitudes and their performance on linear programming problem-solving tasks. This suggests that contextual learning not only improves attitudes but also supports the achievement of learning outcomes (Wakhata et al., 2024). Younger students tend to exhibit more positive attitudes toward contextual learning compared to older students. A study in Australia found that third and fourth grade students showed greater enthusiasm for problem-solving than fifth and sixth grade students, suggesting that the effectiveness of contextual tasks may vary depending on students' developmental stage (Russo & Minas, 2021). Gender may also play a role in attitudes toward contextual learning. Research has shown that boys are more likely to express positive attitudes toward problem-solving tasks compared to girls, although this difference may diminish as students progress through school (Russo & Minas, 2021).

3. Research goal and hypotheses

The goal of this study was to examine whether contextual tasks influence the perceived usefulness of mathematics, and consequently, the positive attitude toward mathematics among first-year students enrolled in professional university studies at the Faculty of Organization and Informatics, University of Zagreb.

Students who perceive mathematics as useful also tend to have greater motivation to learn it. This link between perceived usefulness and motivation has been confirmed by previous research. For example, a study conducted by Gutierrez-Aguilar and colleagues demonstrated a causal relationship between perceived usefulness, student cooperation, behavioral intention, and motivation (Gutierrez-Aguilar et al., 2022). Furthermore, Alassafi (2022) confirmed a positive correlation between motivation and behavioral intention, which is closely related to perceived usefulness and ease of use. Motivation has proven to be a key factor for students, as learning would not be possible without it. Higher levels of motivation are associated with better academic performance (Zohoorian, 2015). Keller (2000) emphasized that motivation is a fundamental component of the learning process, and research by Tsai and Liao (2021) also supports the connection between perceived usefulness and motivation. Based on these findings, the following hypothesis is proposed:

H1: Perceived usefulness has a positive effect on student motivation.

Furthermore, students with higher motivation to learn are generally more satisfied with mathematics. Ha et al. (2019) found that students with stronger motivation to participate in learning activities reported higher levels of satisfaction. Their statistical analyses revealed significant positive effects of motivation on learning satisfaction. Similarly, Yahia et al. (2018) showed that students' initial motivation was a stronger predictor of satisfaction with teachers than academic performance. Based on these findings, the following hypothesis is proposed:

H2: Motivation has a positive influence on student satisfaction.

Finally, students who are more satisfied with learning are expected to have a more positive attitude toward learning mathematics and mathematics as a subject. Research by Manatad and Baluyos (2023) demonstrated that students' satisfaction – particularly with their teachers and interaction during hybrid learning – is significantly related to their academic success and attitudes toward mathematics. They also identified that mathematical aptitude, attitudes, and satisfaction are strong predictors of academic achievement. In addition, research in Tanzania found that students perform better when they enjoy their lessons, highlighting the role of enjoyment in learning mathematics. This suggests that satisfaction derived from enjoyment positively influences students' attitudes toward learning mathematics (Mazana et al., 2019). Based on these findings, the following hypothesis is proposed:

H3: Satisfaction has a positive impact on students' attitudes toward learning mathematics.

To help students perceive mathematics as useful, contextual tasks involving applications in other disciplines such as physics and chemistry were incorporated into teaching materials. The aim of the study was to confirm the positive impact of contextual tasks on perceived usefulness, and consequently, on students' attitudes toward learning mathematics. Based on this, the following hypothesis was formulated:

H4: Contextual tasks have a positive impact on perceived usefulness and thus on attitudes toward learning mathematics.

4. Data analysis

The study utilized a model consisting of five constructs: contextual tasks (CT), motivation (MOT), satisfaction (SAT), perceived usefulness (PU), and attitudes toward learning mathematics (ATL). The constructs MOT, SAT, PU, and ATL were measured using items (manifest variables) MOT1–4, SAT1–5, PU1–3, and ATL1–3, respectively. These items were adapted from existing literature to fit the context of the study. The number of tasks completed (KT1) and satisfaction with tasks (KT2) were used as indicators for the contextual tasks construct. The proposed model is shown in Figure 1.

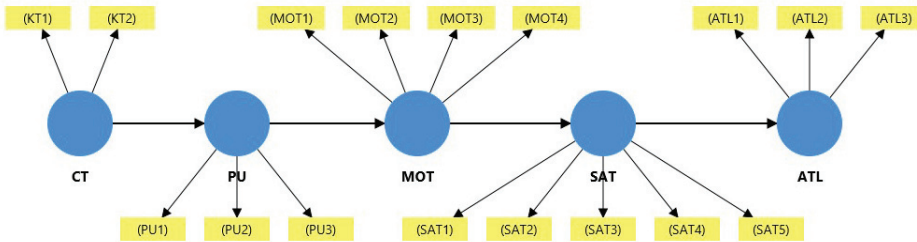

Figure 1. Proposed model.

The research was conducted among first-year students enrolled in professional studies at the Faculty of Organization and Informatics, University of Zagreb. As part of the Mathematics course, students complete contextual tasks within their assignments. A total of four such tasks are included throughout the course.

Empirical data were collected using a questionnaire comprising 17 items. Of these, 16 items were rated on a Likert scale ranging from 1 (completely disagree) to 5 (completely agree). Item KT1 required respondents to select a value from 0 to 4, based on the number of contextual tasks they had completed in the "Assignment" activity, which is part of the student progress monitoring model.

A total of 72 out of 102 students completed the questionnaire. The empirical data were analyzed using the SmartPLS4 software. To ensure the quality of measurement procedures and statistical inference, the reliability and validity of the measurement instrument were assessed. Reliability refers to the consistency and stability of measurement results, while validity indicates whether the instrument accurately measures the intended construct.

Reliability and validity were verified through outer loading analysis using the PLS-SEM algorithm. The analysis indicated that item KT1 (related to the CT construct), item MOT4 (related to the MOT construct), and item ATL2 (related to the ATL construct) should be excluded from the model due to low outer loadings of –0.244, 0.620, and 0.512, respectively. The coding of KT1 was additionally verified and no data-entry errors were identified. The unexpected negative outer loading suggests that KT1 (behavioural indicator) and KT2 (perceptual indicator) do not adequately represent a single reflective construct.

To avoid having a single-item CT construct, the two associated items will be treated as separate manifest variables: satisfaction with contextual tasks (CT2) and the number of contextual tasks solved (CT1), rather than being grouped under a single latent construct. Accordingly, hypothesis H4 has been revised and split into two separate hypotheses:

H4-1: Satisfaction with contextual tasks has a positive effect on perceived usefulness.

H4-2: The number of contextual tasks solved has a positive impact on perceived usefulness.

The revised models are presented in Figure 2.

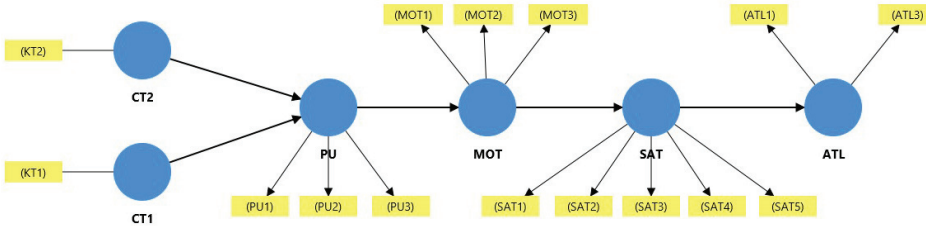

Figure 2. Revised model.

The reliability of the structural model was assessed by analyzing factor loadings, average variance extracted (AVE), and internal consistency of the constructs. As shown in Table 1, all measures in this study have factor loadings above 0.7, and the reliability and AVE values for each construct exceed the thresholds of 0.7 and 0.5, respectively. This indicates that the measurement instruments accurately capture the research constructs and provide reliable and valid conclusions.

Table 1. Indicators of reliability of the structural model.

	Cronbach's alpha	**Composite reliability (rho_a)**	**Composite reliability (rho_c)**	**Average variance extracted (AVE)**
ATL	0.700	0.738	0.867	0.766
MOT	0.876	0.880	0.924	0.801
PU	0.819	0.820	0.892	0.735
SAT	0.824	0.842	0.876	0.587

Discriminant validity is an assessment method used to determine the extent to which constructs are distinct from one another. It demonstrates that items associated with a particular construct show minimal or no association with other constructs in the model. Table 2 presents the heterotrait-monotrait ratio (HTMT), indicating the absence of closely related constructs, as all values are below the 0.85 threshold.

Table 2. Discriminant validity.

	ATL	**SAT**	**MOT**	**PU**	**CT1**	**CT2**
ATL						
SAT	0.557					
MOT	0.550	0.404				
PU	0.732	0.437	0.784			
CT1	0.144	0.051	0.084	0.106		
CT2	0.258	0.260	0.454	0.225	0.202	

Internal model analysis primarily examines the causal effects and significance of relationships between latent variables. This study conducted a significance analysis by estimating path coefficients, p-values, and t-values for the hypothesized relationships. In addition, R^2 values were used to explain the extent of variance accounted for by the proposed relationships between constructs. R^2 ranges from 0 to 1, with values closer to 1 indicating stronger relationships. The results are presented in Table 3.

Table 3. R^2 values for constructs.

	R^2
ATL	0.195
MOT	0.443
PU	0.060
SAT	0.121

To further evaluate the importance of relationships between constructs in the model, a path analysis was conducted by estimating the coefficients and testing their statistical significance. The final results of the structural model analysis are shown in Table 4.

Table 4. Structural model analysis.

Hypothesis	Path	Original sample (O)	Sample mean (M)	Standard deviation (STDEV)	T statistics (|O/STDEV|)	P values
H1	**PU → MOT**	0.665	0.674	0.059	11.284	0.000
H2	**MOT → SAT**	0.348	0.359	0.092	3.780	0.000
H3	**SAT → ATL**	0.441	0.459	0.099	4.477	0.000
H4-1	**CT2 → PU**	0.232	0.236	0.108	2.141	0.032
H4-2	**CT1 → PU**	–0.141	–0.142	0.120	1.173	0.241

5. Discussion of results

Hypothesis H1 examined the relationship between perceived usefulness and student motivation. The results show that the PU construct significantly correlates with MOT ($\beta = 0.665$, t-value $= 11.284$), confirming hypothesis H1.

Hypothesis H2 explored the relationship between student motivation and satisfaction. The results indicate that the MOT construct significantly correlates with SAT ($\beta = 0.348$, t-value $= 3.780$), confirming hypothesis H2.

Hypothesis H3 investigated the influence of satisfaction on attitudes towards learning. As shown in Table 4, the SAT construct significantly correlates with ATL ($\beta = 0.441$, t-value $= 4.477$), confirming hypothesis H3. These findings suggest a chain relationship: perceived usefulness drives motivation, motivation increases satisfaction, and satisfaction positively influences attitudes towards learning mathematics. Perceived usefulness shows a weak influence on other constructs due to its low R^2 value, indicating it is not a strong predictor of attitudes towards learning. In contrast, motivation emerges as the central variable and the main predictor of satisfaction. This relatively low R^2 value indicates that perceived usefulness is influenced by additional factors that were not included in the proposed model. Therefore, the proposed model should be interpreted as explaining only

one part of the factors contributing to perceived usefulness rather than providing a comprehensive explanation of this construct.

Hypothesis H4-1, stating that "Satisfaction with contextual tasks has a positive effect on perceived usefulness," is supported by a significant correlation between CT2 and PU ($\beta = 0.232$, t-value $= 2.141$). This suggests that satisfaction with contextual tasks significantly contributes to the perception of usefulness.

However, Hypothesis H4-2, which tested the influence of the number of contextual tasks solved on perceived usefulness, was not confirmed. The CT1 variable shows a negative and statistically insignificant effect on PU ($\beta = –0.141$, p-value $= 0.241$), leading to the rejection of this hypothesis. Interestingly, although it might be expected that solving more contextual tasks would enhance the perceived usefulness of mathematics, this was not supported by the data in this study. This finding suggests that the number of completed contextual tasks alone may not be sufficient to increase students' perception of the usefulness of mathematics. Students' satisfaction with contextual tasks appears to be more important than the quantity of completed tasks.

6. Limitations

This study has several limitations that should be considered when interpreting the findings. First, the research was conducted on a relatively small sample from a single higher education institution, which limits the generalizability of the results. Second, attitudes toward learning mathematics were measured only once, after the implementation of contextual tasks, without a preliminary (pre-test) measurement. Consequently, changes in students' attitudes over time cannot be directly assessed. Third, the perceived usefulness construct showed relatively low explanatory power ($R^2 = 0.060$), indicating that many additional factors influencing this construct were not included in the proposed model. Finally, the number of completed contextual tasks was not significantly associated with perceived usefulness, suggesting that the effectiveness of contextual tasks depends not only on their completion but also on students' learning experiences.

7. Conclusion

Investigating the relationship between contextual tasks in mathematics teaching and the perceived usefulness of mathematics provides valuable insights into the relationships between contextual learning, motivation, satisfaction, and attitudes toward mathematics. The findings suggest that students' satisfaction with contextual tasks is associated with a higher perception of the usefulness of mathematics. Through increased motivation and learning satisfaction, this relationship may indirectly contribute to more positive attitudes toward learning mathematics. However, these findings should be interpreted with caution due to the limitations of the study. The observed relationships support the inclusion of meaningful contextual tasks

in mathematics instruction as one of the approaches that may enhance students' perception of the usefulness of mathematics and, indirectly, their motivation and learning satisfaction.

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Contact addresses:

Mihaela Bosak
Faculty of Organization and Informatics, University of Zagreb
Pavlinska 2, 42 000 Varaždin, Croatia
e-mail: mihaela.laljek@foi.unizg.hr

Marija Jakuš
Faculty of Organization and Informatics, University of Zagreb
Pavlinska 2, 42 000 Varaždin, Croatia
e-mail: marija.jakus@foi.unizg.hr

Utjecaj kontekstualnih zadataka na stav studenata prema učenju Matematike

Mihaela Bosak i Marija Jakuš

Fakultet organizacije i informatike, Sveučilište u Zagrebu, Hrvatska

Sažetak. Poznato je da kontekstualni zadaci s problemima iz stvarnog života mogu povećati angažman i poboljšati akademska postignuća, ali još uvijek postoji rasprava o tome u kojoj mjeri takav pristup utječe na dugoročne stavove studenata prema matematici. Ovo istraživanje bavi se utjecajem kontekstualnih matematičkih zadataka na stavove studenata prema učenju matematike, s posebnim naglaskom na njihovu percepciju korisnosti matematike, motivaciju i zadovoljstvo učenjem. Cilj istraživanja bio je ispitati odnose između kontekstualnih zadataka, percepcije korisnosti, motivacije, zadovoljstva i općeg stava prema učenju matematike.

Empirijsko istraživanje provedeno je među studentima prve godine stručnog studija na Fakultetu organizacije i informatike Sveučilišta u Zagrebu, pomoću anketnog upitnika, a za analizu prikupljenih podataka korištena je metoda strukturnog modeliranja putem softvera SmartPLS4. Model istraživanja obuhvatio je četiri konstrukta: percipiranu korisnost, motivaciju, zadovoljstvo te opći stav prema učenju matematike. Uz konstrukte, promatrane su i dvije manifestne varijable, zadovoljstvo kontekstualnim zadacima i broj riješenih zadataka.

Rezultati analize pokazali su kako percipirana korisnost značajno utječe na motivaciju studenata, a visoka razina motivacije dovodi do većeg zadovoljstva studenata učenjem matematike. Nadalje, zadovoljstvo učenjem matematičkih sadržaja pozitivno je povezano s općim stavom studenata prema matematici. Analizom prikupljenih podataka, potvrđena je i značajna povezanost između zadovoljstva kontekstualnim zadacima i percipirane korisnosti matematike, što sugerira da studenti, koji rješavaju kontekstualne zadatke s problemima iz stvarnog života, češće percipiraju matematiku kao korisnu. Međutim, rezultati istraživanja nisu pokazali značajnu izravnu povezanost između broja riješenih kontekstualnih zadataka i percipirane korisnosti matematike, kao ni između zadovoljstva kontekstualnim zadacima i općeg stava studenata prema matematici.

Ovi zaključci istraživanja naglašavaju važnost kontekstualnog pristupa u nastavi matematike jer on može poboljšati percepciju korisnosti matematike, a time i povećati motivaciju i zadovoljstvo studenata te neizravno doprinijeti pozitivnijem stavu studenata prema matematici.

Ključne riječi: matematika, kontekstualni zadaci, stav prema učenju, percipirana korisnost, motivacija

Mathematical Personality in the Context of Mathematical Problem-Solving Strategies

Zdenka Kolar-Begović[1,2], Diana Moslavac Bičvić[2] and Ružica Kolar-Šuper[2*]

[1]School of Applied Mathematics and Informatics, Josip Juraj Strossmayer University of Osijek, Croatia
[2]Faculty of Education, Department of Natural Sciences, Josip Juraj Strossmayer University of Osijek, Croatia

Abstract. Students approach mathematical problems using a range of strategies that reflect their individual learning styles and ways of engaging with mathematical situations. Krutetskii established that such preferences form a mathematical cast of mind, and distinguished three types: the analytic, which thinks in verbal-logical terms; the geometric, which thinks in visual-pictorial terms; and the harmonic, which combines the two. Sharma restates this distinction for classroom use under the heading of mathematical personality, contrasting a quantitative with a qualitative type while noting that most individuals display a combined type. Quantitatively oriented students solve tasks methodically, emphasising clarity of procedure and precision in applying standard algorithms, but may struggle with non-routine problems. Qualitatively oriented students approach problem situations holistically, relying on intuition, visual imagery and the creative recognition of relationships between mathematical concepts, while strictly formalised procedures may present a difficulty for them. This paper examines how these orientations appear in the strategies that pupils and pre-service primary teachers apply when solving problem tasks. Written solutions to two non-routine problems were collected from 184 fourth-year students across five consecutive academic years and from 26 fourth-grade pupils with a special interest in mathematics, and each solution was classified as quantitative, qualitative, combined, or answer-only. The aim is to identify, through a qualitative analysis of the solutions, which orientation predominates, and to examine how it relates to the structure of the task. The findings may assist teachers in adapting instructional methods, in fostering deeper mathematical thinking, and in evaluating student work in a way that takes account of the orientation a student brings to it.

Keywords: mathematical personality, problem tasks, problem-solving strategies, quantitative and qualitative thinking, visualisation

*Corresponding author.

1. Introduction

When solving mathematical problems, students draw on different approaches that reflect the way they think, connect information and organise mathematical ideas. These approaches are evident in how a student perceives relations among the given data, how the solution process is planned, and which strategies are deployed when confronting an unfamiliar situation. The set of such characteristics may be regarded as a type of mathematical personality, which indicates not the level of mathematical knowledge but the preferred mode of mathematical thinking and the spontaneous selection of strategies.

Understanding mathematical personality offers teachers deeper insight into student reasoning and contributes to interpreting how students structure their procedures. When a teacher recognises how a student approaches a task, it becomes easier to explain the source of an error, to see what the student understands and what causes difficulty, and to guide the student towards a fuller grasp of mathematical relations. Such insight matters particularly in work with students who show a special interest in mathematics, since they often develop more stable patterns of thinking and more pronounced individual approaches, so that their strategies sometimes depart from the usual patterns of solving.

Mathematical personality may also be situated within the broader framework of theories of mathematical thinking, in which analytic, visual and intuitive approaches are distinguished alongside various heuristic solving strategies. Problem tasks suit such an investigation because they require the solver to select a strategy, carry out a procedure and recognise relations among quantities. Natural preferences become visible precisely in those choices.

According to Sharma (2001), two fundamental types of mathematical personality are distinguished: quantitative and qualitative. The quantitative type is characterised by stepwise, algorithmic solving that relies on known methods, while the qualitative type rests on perceiving relations, structures and patterns with the use of visual representations. In actual, everyday solutions these approaches are frequently interwoven, although one mode of thinking usually stands out more clearly.

This paper analyses the strategies that university students, as well as pupils with a special interest in mathematics, apply when solving two problem tasks, with the aim of identifying features of the quantitative and the qualitative mathematical personality. Through a qualitative analysis of written solutions, the study seeks to establish which strategies participants spontaneously select, how they structure the solution, and in what way differences in mathematical personality are manifested in their thinking. The findings contribute to an understanding of the different forms of mathematical thinking and serve as support for shaping instructional methods that acknowledge the diversity of approaches to mathematics in the pupil and student populations.

The study is guided by two research questions:

1. *Which mathematical personality orientation predominates in the written solutions of pre-service primary teachers and pupils with a special interest in mathematics?*
2. *Does the distribution of mathematical personality orientations in written solutions differ between two problem tasks of different structure, and is the task-related pattern replicated across five consecutive student cohorts?*

2. Mathematical personality

2.1. The concept of mathematical personality

The term mathematical personality denotes the characteristic manner in which an individual perceives, processes and organises mathematical information. In the present study, however, the construct is treated as an instructional framework for interpreting problem-solving strategies proposed by Sharma (1989, 2001), rather than as an established psychological trait. Consequently, the analysis does not attempt to measure stable personality characteristics but instead interprets the orientations reflected in participants' written solutions. Thus, mathematical personality does not describe how much mathematics a person knows, nor how successful that person is; rather, it describes the route by which mathematical meaning is constructed. Sharma (1989, 2001) frames the concept around a distinction familiar from wider research on cognitive style: information can be processed sequentially, element by element, or globally, as an integrated whole. Applied to mathematics, this yields a continuum whose two poles Sharma calls the quantitative and the qualitative mathematical personality.

Sharma is explicit that this is a continuum rather than a dichotomy. Every learner occupies some position along it, and most occupy positions between the extremes. What varies is which pole exerts the stronger pull when a person meets an unfamiliar task and must decide, without external guidance, where to begin. For this reason the construct is best observed not in routine exercises, where the procedure is given, but in problem situations in which the solver must first decide what kind of problem this is.

Sharma's typology is not an isolated proposal. It restates, in the language of classroom practice, a distinction Krutetskii (1976) had established empirically two decades earlier, on the basis of research conducted with mathematically capable schoolchildren between 1955 and 1966. Krutetskii argued for the existence of a mathematical cast of mind and identified three basic types: the analytic, which thinks in verbal-logical terms; the geometric, which thinks in visual-pictorial terms; and the harmonic, which combines the characteristics of the other two. The correspondence with Sharma's scheme is close. Krutetskii's analytic type maps onto the quantitative mathematical personality, the geometric onto the qualitative, and the harmonic onto the combined type. Krutetskii also drew a distinction that Sharma's account depends on without stating it as clearly. The level of mathematical ability, he argued, is determined largely by the verbal-logical component, whereas the type

of ability is determined by the strength of and preference for the visual-pictorial component. Level and type are thus independent, which is what licenses the claim that personality says nothing about how much mathematics a person knows.

This lineage matters for the present study in two respects. It supplies the construct with an empirical foundation independent of Sharma's own practitioner-oriented account, and it connects the construct to an established methodological tradition. Presmeg (1986) operationalised the same distinction through the Mathematical Processing Instrument (MPI), a set of problems each of which admits both a visual and a non-visual solution; the solver's position on a continuum of mathematical visuality is inferred from which route is taken. Presmeg (1986) defines mathematical visuality as the extent to which a person prefers visual methods when attempting problems that can be solved either way. The logic of the present study's instrument is the same, although the implementation is qualitative rather than scored: two problems, each solvable by a computational or a structural route, and a classification derived from the route the participant actually took.

One consequence of that lineage should be stated at the outset, because it governs how the findings below are to be read. An instrument of the MPI type classifies a person across a battery of problems. The present study classifies a solution. These are not the same object. A solver who is harmonic across a battery may well produce a single-route solution to any one problem, and two problems cannot establish a stable personal orientation. What the analysis that follows therefore identifies is strategy selection on two specific problem tasks, which is evidence bearing on mathematical personality rather than a measurement of it. The term is retained because it names the construct under investigation, but the claims are pitched at the level of the solution throughout.

Mathematical personality is related to, but distinct from, several neighbouring constructs. It is not mathematical identity, which concerns how a person sees themselves in relation to mathematics and how that self-perception is negotiated socially (Barba, 2022; Gardee & Brodie, 2023; Ridho et al., 2023). It is not mathematical mindset in Boaler's (2016) sense, which concerns beliefs about the malleability of mathematical ability. Nor is it reducible to affect, meaning the attitudes, beliefs and emotions a learner holds towards mathematics (McLeod, 1992; Hannula et al., 2016). Mathematical personality is narrower and more cognitive than any of these. It concerns the preferred processing route, not the self-concept, the belief system, or the emotional relation to the subject. The distinction matters methodologically, since identity, mindset and affect are typically accessed through self-report, whereas mathematical personality must be inferred from the traces a solver leaves in a written solution.

2.2. Mathematical personality in the context of problem solving

Problem solving is the natural setting in which mathematical personality becomes observable. Pólya (2004) divides the process into four phases: understanding the problem, devising a plan, carrying out the plan, and looking back. Individual preference asserts itself in the second of these. A routine exercise never reaches that

phase in any meaningful sense, because the plan is supplied by the task. A genuine problem, on Schoenfeld's (1985) definition, is one for which the solver possesses no ready-made procedure and must therefore commit to an approach on the basis of how the situation is read.

Schoenfeld (1985) further distinguishes resources, heuristics, control and beliefs as components of problem-solving competence, and locates the decisive differences between solvers not in resources but in control, that is, in the monitoring and regulation of one's own approach. Mathematical personality can be understood as the default setting of that control system, the approach a solver adopts before any monitoring has taken place. Mason et al. (2010) describe the same moment through the phases of entry, attack and review, and note that it is at entry that the solver's habitual way of seeing exerts its strongest influence.

This is why non-routine tasks with a clear internal structure are particularly diagnostic. Where a task admits both a computational route and a structural route to the same answer, the route taken is informative. Where a task admits only one route, it reveals procedural competence but says little about personality.

2.3. The quantitative mathematical personality

The quantitative mathematical personality processes information sequentially and procedurally, step by step. Sharma (2001) characterises this learner as one who seeks a known algorithm and applies it consistently, favouring clearly defined steps, arithmetic manipulation and the linear organisation of the solution. Presented with a task, such a solver looks for the type of the task, then for the method appropriate to that type, and then executes it.

The strengths of this orientation are considerable: precision, reliability, transparency of procedure, and success in arithmetic and algebra, where the domain rewards exactly these dispositions. Sharma notes that such learners are typically successful in algebra and in those parts of geometry that rest on systematic procedures and axioms.

The quantitative orientation should not be equated with weak mathematical ability. Krutetskii (1976) placed the analytic type among the mathematically capable, and characterised capable pupils generally by three features: the rapid generalisation of mathematical material, the curtailment of reasoning once a structure has been grasped, and the striving for an elegant solution. These are properties of ability rather than of type, and they may be found at either pole. The quantitative type describes the route a solver takes, not the quality of the traveller.

The characteristic weakness appears when no known algorithm applies. Faced with a genuinely novel situation, the quantitative solver may attempt to force it into the nearest familiar template, or may become dependent on the recall of several methods without being able to choose among them. Sharma reports the typical utterance of such a learner as an appeal to be shown the method for solving that kind of task. In problems whose difficulty lies in the interpretation of a relation rather

than in computation, a strictly quantitative approach produces a correct execution of a wrong plan.

2.4. The qualitative mathematical personality

The qualitative mathematical personality processes mathematical information primarily visually and holistically, moving from the whole towards the parts. Sharma (2001) describes such a learner as exploring global solution routes, comparing and connecting different concepts, and beginning by playing with the problem before arriving at a solution. Presmeg (1986) documents the same orientation under the heading of visualisation, showing that visual thinkers in mathematics construct and manipulate mental images as their primary means of reasoning, and that this style is neither a deficiency nor an automatic advantage but a distinct route to mathematical understanding.

Such solvers make rich use of spatial reasoning, sketches, diagrams and models; they often perceive the answer intuitively before they can justify it, and they may resist writing the procedure down because the insight arrived whole. Their strength lies in geometry, in problems of pattern and structure, and in situations where recognising a relation dissolves the computational work altogether.

An important qualification follows from Presmeg's three-year classroom study, reviewed in Presmeg (2006), which bore out Krutetskii's observation that students who possess the ability to use visual methods may nonetheless prefer not to do so. Preference and capacity are distinct, and preference is shaped by what instruction has rewarded. A learner who has been taught that a solution counts only when it is written as a sequence of computations may suppress a visual route that would have served better. This has a direct methodological consequence for the present study, which infers personality from written products: what is recorded reflects preference under the perceived demands of the setting, not necessarily the full repertoire available to the solver.

The characteristic weaknesses mirror those of the quantitative type. Procedural work may feel tedious and be experienced as an interruption of thinking; steps may be omitted from the written record as self-evident; arithmetic slips creep into an otherwise correct line of reasoning; and where automatisation of computation is insufficient, the solver may lose the thread of the argument while managing the numbers. Sharma also observes that in the extreme case such a learner may fail to master computation adequately, and thus be unable to demonstrate what has in fact been understood.

2.5. The combined type

Most people display a combined type. Sharma (2001) is emphatic that the two orientations are not mutually exclusive and that the majority of learners use both approaches, with one usually dominating. The combined type is not a midpoint of weakness but, on the contrary, the profile associated with mature mathematical

competence: the good mathematician sees both aspects of a problem and unites both styles of learning within themselves.

Kilpatrick et al. (2001) reach the same conclusion by an independent route. Mathematical proficiency, in their account, consists of five intertwined strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. Proficiency requires all five. A purely quantitative profile over-develops procedural fluency at the expense of strategic competence, and a purely qualitative profile risks the reverse. The pedagogical target is accordingly not the conversion of one type into the other, but the enrichment of each with the strengths of the other.

2.6. The importance of recognising mathematical personality

Sharma's (2001) practical claim is that mathematical personality is the factor to be established before choosing an instructional or therapeutic procedure, since a student will make considerably greater progress when the mode of teaching corresponds to their style of learning. Four consequences for practice follow.

First, diagnosis of error. A wrong answer produced by a quantitative solver who applied a well-executed but inappropriate algorithm calls for a different intervention from a wrong answer produced by a qualitative solver whose reasoning was sound but whose arithmetic slipped. Without recognising the personality type, both appear simply as errors.

Second, assessment. Written work is conventionally rewarded for procedural completeness, which systematically privileges the quantitative type. A qualitative solution that arrives at a correct insight but omits the intermediate steps may be marked as incomplete, even though it evidences the deeper mathematical understanding. Recognising the type permits a fairer reading of what the work demonstrates. A concrete instance of such assessment, on a task introduced in Section 3.3, is discussed in Section 5.

Third, task design. If the tasks a teacher sets activate only the quantitative route, the qualitative repertoire is neither exercised nor visible. A balanced set, containing computational and structural problems alike, is a precondition for developing the combined type.

Fourth, teacher education. Pre-service teachers teach as they themselves think unless made aware of the alternative. A pre-service teacher whose own mathematical personality is strongly quantitative and who has never had that fact named will tend to model only quantitative approaches, and to read qualitative solutions from their future pupils as deficient. This is the specific reason the present study takes fourth-year pre-service primary teachers as its principal sample.

3. Methodology

3.1. Research design

The research is of an applied character and employs a qualitative content analysis of written solutions, with descriptive quantification of the resulting categories; the frequencies reported below are used descriptively only, and no inferential claims are made. Written solutions to two problem tasks were analysed in order to identify the strategies participants applied when solving them. In the student group, a repeated cross-sectional design was used, as data were collected from five independent cohorts over five consecutive academic years. This made it possible to examine whether the observed task-related pattern was replicated across independent cohorts at the same level of education. In the pupil group data were collected at a single point of measurement in order to gain insight into the problem-solving strategies of pupils with a special interest in mathematics. The sample was selected by convenience, taking into account the availability of participants in the instructional and extracurricular settings in which the research was conducted.

3.2. Participants

The student group comprised a total of 184 fourth-year students of the Integrated Undergraduate and Graduate Teacher Education Programme, of whom 45 participated in the academic year 2021/2022, 33 in 2022/2023, 40 in 2023/2024, 30 in 2024/2025, and 36 in 2025/2026. The selection of fourth-year students was deliberate, since at that stage of study they already possess the necessary prior mathematical knowledge and are developing methodological competences, which makes them suitable participants for research directed at the analysis of problem-solving strategies.

The research also involved 26 fourth-grade pupils from primary schools in Osijek. The pupils had previously been identified as having a special interest in mathematics based on their teachers' recommendation, regular participation in supplementary mathematics instruction, or participation in mathematics competitions. They took part in the workshops of the Little Mathematics School, which allowed the tasks to be administered in a stimulating environment under equal conditions for all participants. The Little Mathematics School at the Faculty of Education is an outreach programme for fourth-grade primary school pupils with a particular interest in mathematics, designed to foster mathematical thinking and problem-solving skills through mathematics learning activities planned and implemented collaboratively by faculty members and pre-service primary school teachers.

3.3. Instruments

Participants were given two problem tasks.

Task 1 (adapted for this study from the classic climbing problem). *"A snail is climbing a column 10 metres high. During the day it climbs 5 metres, and during the night it slides down 4 metres. How many days does it need to reach the top of the column?"*

The correct solution is six days. The reasoning rests on the fact that the snail's net daily progress is one metre. By the end of the fifth day, before the night slide, the snail is at 9 m; after the fifth night it is at 5 m. On the sixth day it climbs the remaining 5 m and reaches the top of the column before sliding back down.

Task 2 (Presmeg, 1989, as reproduced in Sharma, 2001, p. 51). *"A dog was running after a fox that was 30 metres away from it. The dog's leap is 2 metres and the fox's leap is 1 metre. While the fox makes three leaps, the dog makes two. How many metres must the dog run to catch the fox?"*

The correct solution is 120 metres. In the time of one repetition of the movement the dog covers 4 metres (2×2 m) and the fox covers 3 metres (3×1 m), so the gap narrows by one metre per cycle. Since the initial distance is 30 metres, 30 cycles are required, in which the dog covers a total of 120 metres (30×4 m).

The two tasks differ in origin. The first is an adaptation of the classic climbing problem, whose oldest known form is Fibonacci's "lion in the pit" in the Liber Abaci of 1202, a variant of the familiar frog- or snail-in-the-well puzzle (Sigler, 2002); it was adapted for this study in its numerical values (the height of the column and the daily climb and nightly slide). The problem has a long history of eliciting erroneous solutions even from distinguished mathematicians, and remains widely used in puzzle collections and textbooks; Cenobio Castillo et al. (2017) review this history and report an empirical study of students' performance on it, which makes it a well-documented instrument for the present purpose. Working with the same historical error, Sliško (2020) treats a solver's mistake as something to be examined rather than merely marked wrong: reconstructing how an erroneous answer comes about can shed light on the approach a solver has taken, so that even an incorrect solution may carry diagnostic value. This resonates with the decision, adopted throughout the present study, to attend to the route a solution takes rather than to its correctness alone. The second task is the problem presented by Presmeg (1989) and reproduced by Sharma (2001, p. 51) as an illustration of the two mathematical personality types at work on a single problem.

Beyond their origins, the two tasks were chosen because they differ in the structural feature that Section 2.2 identifies as diagnostic. Task 1 presents a situation that looks linear but is not, its difficulty residing in a single boundary condition at the final day, where the regular pattern breaks. Task 2 presents a dynamic relation between two quantities changing at once, its difficulty residing in the recognition of a constant differential. Both are non-routine in Schoenfeld's (1985) sense for both populations, and each admits a computational and a structural route to the same answer.

3.4. Procedure

Students and pupils solved the tasks independently, in written form, without the use of additional materials or mutual consultation. The students solved the tasks during classes within the course Teaching Mathematics II, while the pupils solved the same tasks during the workshops of the Little Mathematics School, under equal organisational and temporal conditions.

3.5. Data analysis

The collected solutions were analysed by a qualitative procedure based on the recognition of features of the types of mathematical personality according to Sharma (2001). Each solution was assigned to one of four categories devised for the purpose.

Category 1 — quantitative approach: solutions distinguished by algorithmic procedures, the linear sequencing of steps and an emphasis on computational methods.

Category 2 — qualitative approach: solutions in which features of the qualitative approach predominated, for example the use of sketches, visual representations, or the intuitive recognition of relations among quantities. A verbal account was coded here only where it described the structural relation holistically; a verbal narration of a sequential computation was coded as quantitative, since verbal-logical processing is, in Krutetskii's terms, the signature of the analytic rather than the geometric type.

Category 3 — combined type: solutions in which participants combined elements of both approaches.

Category 4 — answer only: solutions in which only the final answer was given without a display of the procedure, so that the type of mathematical personality could not be determined.

Each solution was coded independently by the three authors against a written protocol specifying indicators for each category, each author working without access to the classifications made by the others. Initial disagreements were confined mainly to the distinction between combined solutions and predominantly quantitative solutions containing incidental visual elements; all disagreements were resolved through discussion until consensus was reached. This classification made it possible to systematically track differences in the approach to problem solving across the collected solutions and to reach a deeper understanding of how mathematical personality is manifested in the actual process of mathematical thinking.

4. Results

The analysis of the written solutions to the two problem tasks provided insight into the way in which, according to Sharma (2001), the types of mathematical

personality are manifested in real problem-solving situations among students of teacher education and pupils with a special interest in mathematics. The results are presented separately for students and for pupils, each group being considered within its own sample. Particular attention is paid to differences between student generations.

4.1. Students: distribution of mathematical personality types

In both tasks a dominant quantitative mathematical personality is observed among the students. One possible explanation for the predominance of quantitative orientations is that students' prior schooling may have placed greater emphasis on computational operations, formalised algorithms, and the consistent application of rules and formulas. However, the two analysed tasks prompted different ways of thinking: the first task, which appears linear and stepwise, more often activated exclusively operational solutions, while the second task afforded room for combining qualitative insight with formal computation.

Table 1. Distribution of mathematical personality orientations in students' solutions ($N = 184$).

Task	Quantitative	Qualitative	Combined	Answer only
Task 1	106 (57.6%)	43 (23.4%)	26 (14.1%)	9 (4.9%)
Task 2	85 (46.2%)	37 (20.1%)	58 (31.5%)	4 (2.2%)

The results for the first task show that the majority of students reach for a quantitative approach to solving, relying on computational procedures, standard algorithms and the stepwise execution of operations. This approach reflects Sharma's (2001) understanding of the quantitative mathematical personality, in which the emphasis falls on numerical manipulation, clearly defined steps and the linear organisation of the solution. Qualitative approaches appear considerably more rarely and most often involve recognising relations among numbers, visualising the situation, or intuitively perceiving the structure of the task. The combined type occurs in cases where students combine computation with a partial insight into the relations among quantities, for instance when they use a computational procedure to confirm a pattern they had previously noticed.

The relatively small share of qualitative and combined solutions suggests that the majority of students perceived the task primarily as a computational problem, and not as a problem in which it is necessary first to interpret the relations and only then to apply the operations. The examples shown in Figure 1, Figure 2, and Figure 3 clearly illustrate the differences between the quantitative and the qualitative approach in this task, in which the key characteristics of each type of mathematical personality are visible. The most common error in the first task was to divide the height of the column mechanically by the net daily progress: since the snail climbs 5 m by day and slides 4 m by night, students reasoned that it advances 1 m per day and therefore needs 10 days for a 10-metre column ($5 - 4 = 1; 10 : 1 = 10$). This

overlooks the boundary condition of the final day, when the snail reaches the top before the night-time slide, so the correct answer is six days.

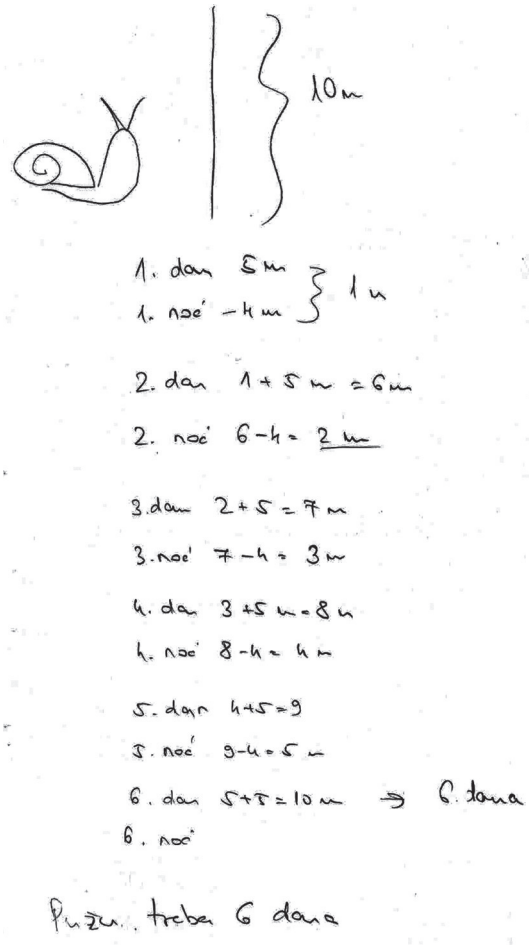

Figure 1. A quantitative solution to the first task.

In Figure 1 the solver proceeds day by day in a vertical column of arithmetic (1st day: $5-4=1$; 2nd day: $1+5-4=2$; ...), reaching the answer through the sequential execution of operations (the drawing plays no part in the solving procedure) – the diagnostic signature of the quantitative route. The units of measurement are not recorded consistently. The solver attaches them to some quantities (5 m, 4 m, 10 m) but omits them elsewhere in the computation – for instance on day 5 ($4+5=9$) and in the entry for night 5 ($9-4=5$ m), where the unit appears with the result but not with the operands. The unit is also attached asymmetrically within a single operation ($1+5$ m $=6$ m), although both addends denote a length. This does not affect the correctness of the result, since all quantities are expressed in the same unit.

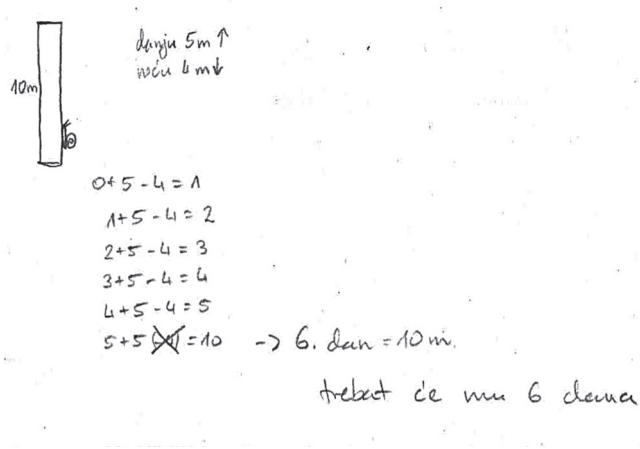

Figure 2. A quantitative solution to the first task, with an initial sketch.

In Figure 2, the route is the same, though it starts differently. Although the solver begins by sketching the column and noting the rule of movement (5 m by day, 4 m by night), the solution is reached entirely by computation – summing the net daily progress day by day ($0 + 5 - 4 = 1$; $1 + 5 - 4 = 2$; ...) and, for the final day, omitting the night-time slide ($5 + 5 = 10$) to conclude that six days are needed. The drawing serves only as an initial orientation (the drawing plays no part in the solving procedure), while the solving procedure itself is sequential and operational – characteristic of the quantitative type of mathematical personality.

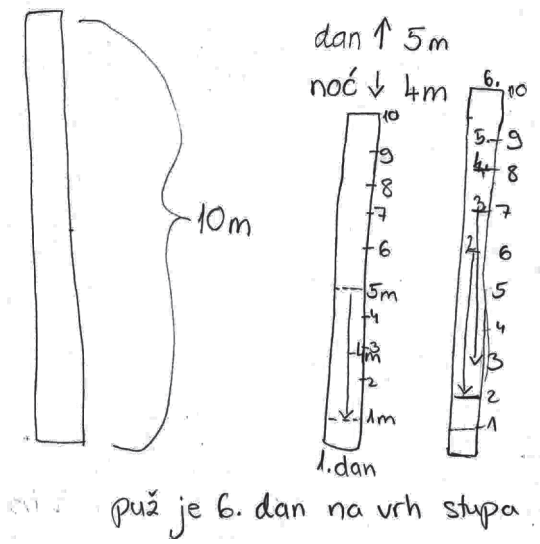

Figure 3. A qualitative solution to the first task.

In Figure 3, the solver represents the column on a vertical scale and draws the day climbs and night slides as arrows along it, reading the moment the snail reaches

the top off the diagram – reasoning that is spatial and holistic rather than computational, the signature of the qualitative route. The solution illustrates how structural insight eliminates the need for sequential computation. The solver represents the process as a spatial transformation rather than an arithmetic calculation.

In the second task the structure of responses shows a different distribution of mathematical personality types. In contrast to the first task, here the share of combined approaches is greater and the share of exclusively quantitative approaches is smaller (Table 1). This shift suggests that the task demanded a deeper conceptual understanding of the relations between quantities, in which students had to interpret the change, recognise the constant difference or the pattern, and connect these observations with computational procedures. Combined approaches most often involve the perception of a pattern or relation followed by verification through computation. Qualitative solutions appear in situations where students build the solution on the basis of the logical structure of the task and on visualisation, without the need for extensive computation.

This situation indicates that students change strategy depending on the nature of the problem task: when a task involves a dynamic relation, a difference or a change, the likelihood of using qualitative and combined strategies increases. The examples shown in Figure 4 and Figure 5 further confirm these differences, since they clearly display the characteristic patterns of quantitative and qualitative solving of the second task.

In Figure 4, the solver tabulates the two positions (dog and fox) in parallel columns and advances them cycle by cycle until they coincide at 120 metres.

In Figure 5, the solver represents the thirty cycles of pursuit as a sequence of drawn units, with a legend establishing that each mark corresponds to one metre, and traces them until the two positions coincide. Although the final numerical calculation ($30 \cdot 4 = 120$) is included, it confirms and quantifies a solution that has already been established through a visual representation of the relationship between the dog and the fox. The count of thirty cycles is read off the drawing rather than derived independently, which distinguishes this solution from the combined type. The dominant problem-solving strategy is therefore qualitative, and the solution was classified in Category 2.

As shown in Figure 6, the student first represented the problem using a schematic drawing, indicating the initial distance between the dog and the fox and the lengths of their leaps. Based on this representation, the student recognised that in each move the dog covered 4 m, whereas the fox covered 3 m, reducing the distance between them by 1 m. The student then tracked the remaining distance after each move (30 m, 29 m, 28 m, ..., 0 m) until the dog caught up with the fox. Finally, the student multiplied the dog's distance per move (4 m) by the required number of moves (30), concluding that the dog had to run 120 m. The drawing thus serves to establish the structure of the relation, but it is abandoned once the numerical rule is available, and the remainder of the solution proceeds computationally. This shift between registers, rather than the mere presence of both a drawing and a calculation, is what marks the solution as combined.

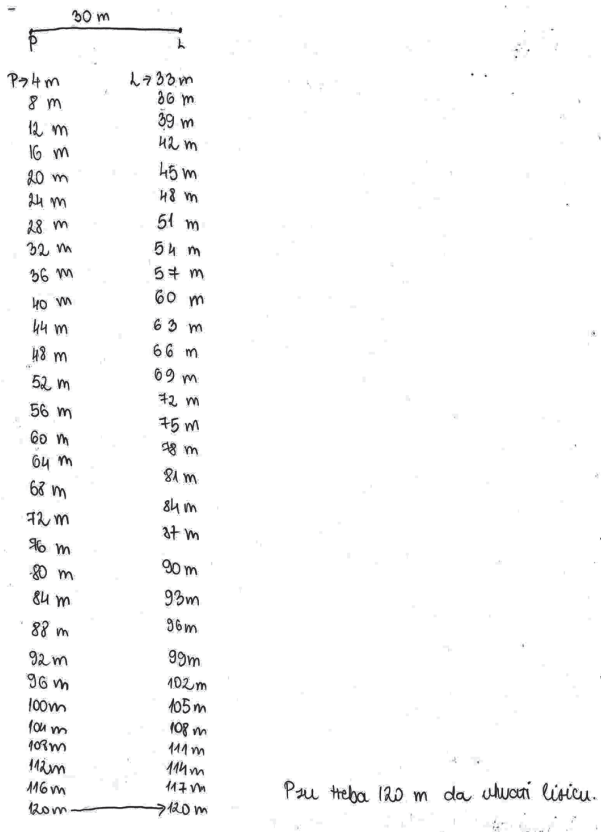

Figure 4. A quantitative solution to the second task.

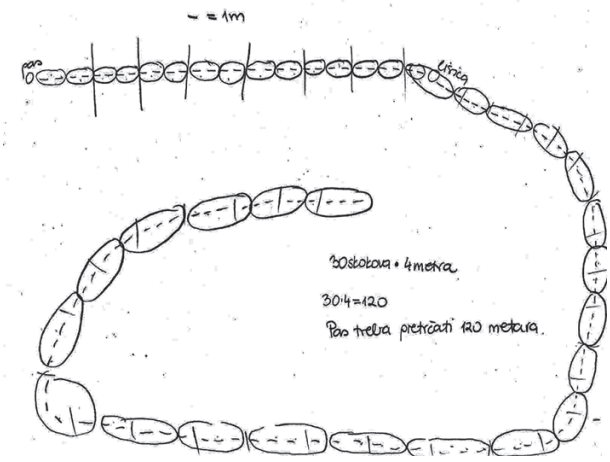

Figure 5. A qualitative solution to the second task.

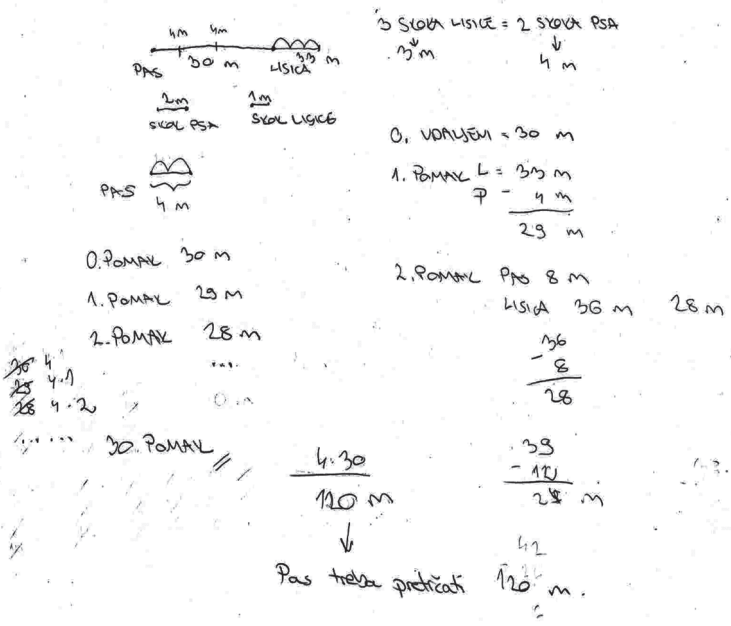

Figure 6. A combined solution to the second task.

4.2. Students: success in solving

Table 2. Accuracy of students' solutions across the two problem tasks ($N = 184$).

Task	Incorrect	Correct
Task 1	151 (82.1%)	33 (17.9%)
Task 2	132 (71.7%)	52 (28.3%)

The rate of correct solutions is strikingly low in both tasks. In Task 1, only 33 students (17.9%) reached the correct answer of six days; in Task 2, 52 students (28.3%) reached the correct answer of 120 metres. The dominant error in Task 1 is the mechanical division of the height of the column by the net daily progress, which yields ten days and overlooks the boundary condition of the final day. Errors in the second task took two forms. Alongside the dominant error of failing to recognise the constant differential of one metre per cycle, errors of execution were also recorded: a faulty step in the computation, most often in repeated addition or in determining the total distance covered, and inconsistent tracking of the two positions in a tabular representation, where the columns for the dog and the fox became confused at some step or one of the sequences was broken off before the point of meeting.

It is worth noting one feature of the tasks themselves: the task with the higher share of combined approaches is also the task with the higher rate of correct solu-

tions. No causal link follows from this, since the two tasks differ in more than one respect, but the pattern is consistent with Schoenfeld's (1985) account of strategic flexibility as the discriminating factor in problem-solving success. Both tasks punish the exclusive application of a single processing route. Task 1 penalises the purely quantitative route, which produces an arithmetically correct division that nonetheless misses what the task asks. Task 2 defeats the purely quantitative route as well, which computes without a model of the relation, and equally punishes a purely intuitive route unsupported by computation.

4.3. Students: stability of the pattern across five generations

Because the student data were collected over five academic years, it is possible to ask whether the pattern just described holds across cohorts or is an artefact of one particular group. Table 3 and Figure 7 set out the distribution by generation for both tasks.

Table 3. Distribution of mathematical personality orientations in students' solutions by generation and task ($N = 184$).

Generation	Task	Quantitative	Qualitative	Combined	Answer only
2021/2022 ($N = 45$)	1	32 (71.1%)	9 (20.0%)	4 (8.9%)	0 (0.0%)
	2	27 (60.0%)	6 (13.3%)	10 (22.2%)	2 (4.4%)
2022/2023 ($N = 33$)	1	15 (45.5%)	9 (27.3%)	9 (27.3%)	0 (0.0%)
	2	11 (33.3%)	12 (36.4%)	10 (30.3%)	0 (0.0%)
2023/2024 ($N = 40$)	1	24 (60.0%)	6 (15.0%)	4 (10.0%)	6 (15.0%)
	2	15 (37.5%)	10 (25.0%)	15 (37.5%)	0 (0.0%)
2024/2025 ($N = 30$)	1	15 (50.0%)	10 (33.3%)	4 (13.3%)	1 (3.3%)
	2	18 (60.0%)	5 (16.7%)	7 (23.3%)	0 (0.0%)
2025/2026 ($N = 36$)	1	20 (55.6%)	9 (25.0%)	5 (13.9%)	2 (5.6%)
	2	14 (38.9%)	4 (11.1%)	16 (44.4%)	2 (5.6%)
Total ($N = 184$)	1	106 (57.6%)	43 (23.4%)	26 (14.1%)	9 (4.9%)
	2	85 (46.2%)	37 (20.1%)	58 (31.5%)	4 (2.2%)

In every cohort, the quantitative orientation predominated in Task 1, and in every cohort the proportion of combined solutions was higher in Task 2 than in Task 1. Although the proportions varied across cohorts, this task-related pattern was replicated in all five cohorts. Given the relatively small cohort sizes and the descriptive design, individual cohort differences should not be overinterpreted. The 2024/2025 cohort produced the highest proportion of qualitative solutions in Task 1 (33.3%), whereas the 2025/2026 cohort showed the highest proportion of combined solutions in Task 2 (44.4%).

Two cautions apply to reading Table 3. The five generations are independent cohorts rather than the same individuals observed repeatedly, so nothing here speaks to how any individual's approach develops over time. And the design contains no

mechanism by which a cohort difference could be attributed to a cause, since the cohorts differ in whatever else changed between one year and the next.

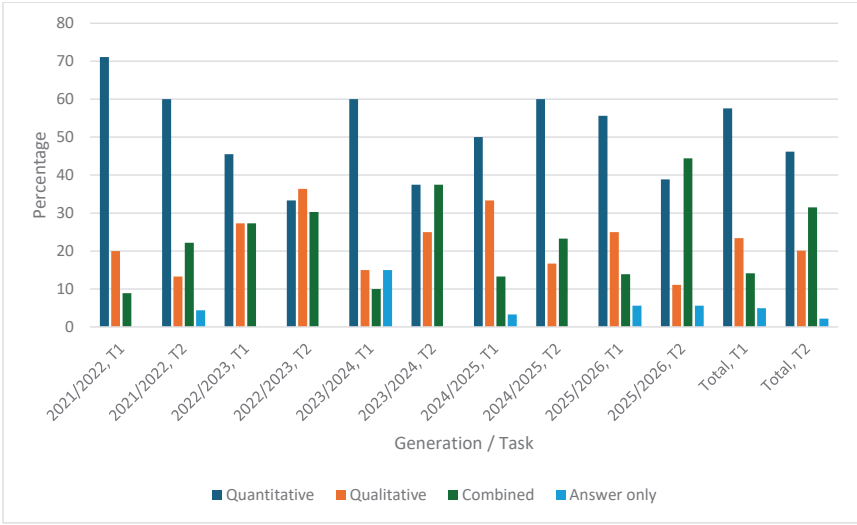

Figure 7. Distribution of mathematical personality orientations in students' solutions by generation and task.

4.4. Pupils with a special interest in mathematics

Pupils with a special interest in mathematics were analysed as a separate unit. Among the pupils, too, the quantitative type of mathematical personality predominates, which is in accordance with their age and the developmental stage of their mathematical thinking.

Table 4. Distribution of mathematical personality orientations in pupils' solutions ($N = 26$).

Task	Quantitative	Qualitative	Combined	Answer only
Task 1	18 (69.2%)	4 (15.4%)	1 (3.8%)	3 (11.5%)
Task 2	16 (61.5%)	2 (7.7%)	5 (19.2%)	3 (11.5%)

In the first task, the pupils most often repeat the procedure without considering the structure of the process. In the second task, some pupils recognise the relation between the movements of the dog and the fox, although only one reached the correct answer by recognising the constant difference in the distance covered. The success of the pupils is shown in Table 5.

Table 5. Accuracy of pupils' solutions across the two problem tasks ($N = 26$).

Task	Incorrect	Correct
Task 1	23 (88.5%)	3 (11.5%)
Task 2	25 (96.2%)	1 (3.8%)

The pupils' pattern differs from the students' in two respects. The dominance of the quantitative approach is more pronounced (69.2% and 61.5%), and combined solutions are correspondingly rarer, appearing only once in Task 1. The pattern also suggests that the special interest in mathematics by which these pupils were identified is not in itself an indicator of a qualitative or flexible orientation. The success rates confirm the difficulty of the tasks for this age group: three pupils solved Task 1 and one pupil solved Task 2 correctly. Given these frequencies, the pupil data support descriptive observations only.

5. Discussion

The quantitative approach is the most frequent single orientation among both students and pupils, in both tasks. Its predominance is, however, weaker in the second task, particularly among students. This task, which presents a dynamic relation between two quantities changing simultaneously and requires recognising a constant differential, appears to have encouraged the use of qualitative elements, mostly in combination with computation: more than half of the students' solutions contained qualitative elements, either alone or combined with computation. Boaler (2016) stresses that tasks with a clearly expressed structure encourage more flexible and creative mathematical thinking, which is consistent with the increased share of combined approaches in our sample. Schoenfeld (1985) further draws attention to the importance of strategic adaptability in problem solving, in which those most successful in solving problem tasks combine heuristic procedures, intuitive insights and formal strategies. It is precisely such flexibility that is manifested in the combined type of mathematical personality observed in our solutions.

Set against Krutetskii's (1976) typology, what stands out in this distribution is the scarcity of the combined type. Studies that classify pre-service teachers with Presmeg's Mathematical Processing Instrument (see Presmeg, 1986, 2006) have repeatedly found the harmonic type to be the most commonly adopted in that population. Our students do not show that profile in Task 1, where combined solutions account for 14.1%, and approach it only partly in Task 2, at 31.5%. The most likely explanation is the difference in unit of analysis set out in Section 2.1. Instruments of the MPI type classify a person across a battery of problems; we classify a solution, so our figures are not directly comparable with that literature.

The sample gives this finding a particular weight. These are fourth-year students who will shortly be teaching problem solving to children of the same age as the pupils in our second group. That 82.1% of them did not solve Task 1, a non-routine task of this kind, warrants attention, though it invites caution in interpretation. A convenience sample and two tasks do not license a conclusion about teacher preparation in general, and the failure rate depends also on the difficulty of the tasks themselves, which were non-routine for both populations. The shape of the failure matters as much as its rate. The modal error was a well-executed algorithm that answered a question the task had not asked, which is the characteristic failure mode of the quantitative orientation as Sharma describes it. Sharma's practical claim is that instruction works better when it corresponds to the learner's style. If that claim holds, it would follow that a teacher whose own repertoire

runs in one channel only is poorly placed to recognise the other channel in a pupil, still less to cultivate it – though this consequence is a hypothesis suggested by the present data rather than a finding established by them.

One documented instance illustrates the mechanism vividly. A fourth-grade primary pupil solved Task 1 visually – drawing the column on a vertical scale and marking the daily movements along it – and correctly concluded that the snail needed six days (Figure 8). The solution was correct, yet in assessing it for selection to a competition the teacher rewrote it as her own arithmetic procedure, day by day, annotated it with the question "PROCEDURE?", and awarded one of a possible ten marks with the comment that it was "mere drawing". The correct qualitative route was not marked wrong but devalued, because it did not match the procedural form the assessor expected. The choice of task was itself apposite: a non-routine problem of this kind separates the quantitative from the qualitative approach, and so served the purpose of selecting pupils well. It is precisely for this reason that the mis-scoring of a correct qualitative solution is the more telling – it stems not from the task but from the assessor's expectation of what a solution should look like. The instance is a single case and lies outside the systematic sample; it says nothing about the frequency of the phenomenon, but shows what it looks like at the moment of assessment.

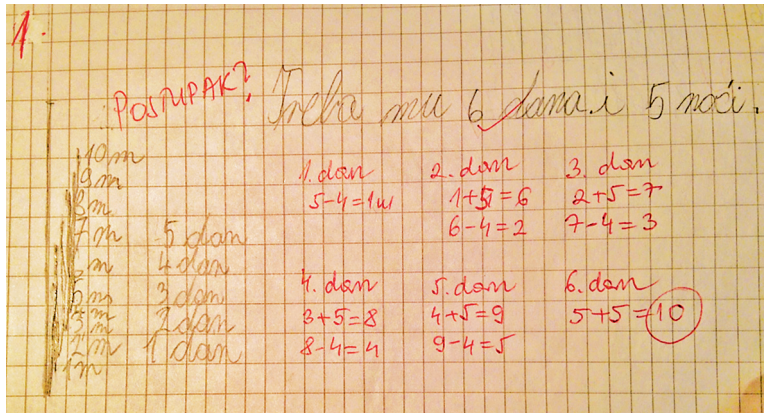

Figure 8. A qualitative solution by a pupil to Task 1 (pencil) and the assessor's quantitative correction (red).

Note. This example is taken from a pupil outside the study sample and is included solely to illustrate how a qualitative solution may be assessed using quantitative criteria.

The pupil results invite one further connection. Our pupils were identified as having a special interest in mathematics, and Krutetskii (1976) worked precisely with mathematically capable children, among whom he found all three types represented. Their overwhelmingly quantitative profile (69.2% and 61.5%) cannot therefore be put down to capability. Age is the more plausible explanation, since coordinating two routes presupposes a repertoire in both, and fourth-graders have had less opportunity to build one. The selection mechanism may also play a part: recommendation, supplementary instruction and competition all tend to reward

procedural fluency, which is the quantitative profile. Krutetskii's separation of type from level is what makes this reading available, since on his account a pupil may be highly capable and strongly analytic at the same time.

Several limitations qualify these findings. The sample was a convenience sample drawn from the settings available to the researchers and is not representative of the wider population of pre-service teachers or of primary school pupils; the findings are therefore not generalisable beyond the groups described.

The two tasks are a narrow instrument, and this is the study's principal limitation. Mathematical personality is a general orientation, but the evidence here comes from two problems that resemble each other closely: both are non-routine word problems about motion, both turn on a repeated cycle, and both conceal their difficulty in a single structural feature. Because the distribution depends on the task in this way, the claims throughout are framed at the level of the solution rather than the person.

6. Conclusion

The analysis of the solutions to the given problem tasks shows that mathematical personality, according to the concept described by Sharma (2001), is manifested through the specific strategies and procedures that participants choose when solving problem situations. In both groups, the features of the quantitative mathematical personality are predominantly present, which points to a strong reliance on standard computational procedures and clearly defined algorithms. However, in solutions requiring the recognition of relations, patterns or changes, qualitative and combined approaches are also identified, especially in the second task.

These findings suggest that the nature of the problem substantially directs the kind of strategies that will be activated, which accords with Boaler's (2016) interpretations of the role of task structure in encouraging different forms of mathematical thinking. The results indicate that mathematical personality is manifested not only in the final answer but also in the manner of organising the procedure, in the choice of strategy, and in the argumentation of the solution.

In this sense the research has offered a valuable empirical insight into the differences among the quantitative, qualitative and combined types of mathematical personality, and has shown how individual types are manifested in the written solving of problem tasks and in the way pupils and students organise their mathematical thinking. The findings are useful in planning instruction, since they can help teachers to interpret students' procedures more precisely and to design tasks that encourage deeper understanding and more flexible mathematical thinking.

Several directions for further work follow. The development of mathematical personality across the years of education would repay systematic monitoring, as would the influence of different didactic approaches on the strengthening of qualitative and combined strategies. Pupils with a special interest in mathematics deserve particular attention, since their solutions often show a sensibility for relations and structures that a competition-oriented curriculum may not reward. Methodologically, the priority is to administer a validated instrument such as the Mathematical

Processing Instrument (see Presmeg, 1986, 2006) alongside a task-based classification with the same participants, which would test whether the two converge and would make findings of this kind commensurable with the wider literature on analytic, geometric and harmonic types. Given the low success rates recorded here, a further question is whether making these orientations explicit within methodology courses, so that students identify their own dominant route and deliberately practise the other, changes the strategies they go on to model in their own classrooms.

The concept of mathematical personality may provide a useful interpretative framework for shaping instruction that acknowledges different forms of mathematical thinking and encourages the development of conceptually rich, flexible and meaningful mathematical understanding.

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Contact addresses:

Zdenka Kolar-Begović
School of Applied Mathematics and Informatics
Josip Juraj Strossmayer University of Osijek
Trg Ljudevita Gaja 6, 31000 Osijek, Croatia
Faculty of Education
Josip Juraj Strossmayer University of Osijek
Cara Hadrijana 10, 31 000 Osijek, Croatia
e-mail: zkolar@mathos.hr

Diana Moslavac Bičvić
Faculty of Education
Josip Juraj Strossmayer University of Osijek
Cara Hadrijana 10, 31 000 Osijek, Croatia
e-mail: dmoslavac@foozos.hr

Ružica Kolar-Šuper
Faculty of Education
Josip Juraj Strossmayer University of Osijek
Cara Hadrijana 10, 31 000 Osijek, Croatia
e-mail: rkolar@foozos.hr

Matematička osobnost u kontekstu strategija rješavanja matematičkih problema

Zdenka Kolar-Begović[1,2], Diana Moslavac Bičvić[2] i Ružica Kolar-Šuper[2]

[1]Fakultet primijenjene matematike i informatike,
Sveučilište Josipa Jurja Strossmayera u Osijeku, Hrvatska
[2]Fakultet za odgojne i obrazovne znanosti, Odsjek za prirodne znanosti,
Sveučilište Josipa Jurja Strossmayera u Osijeku, Hrvatska

Sažetak. Studenti pristupaju matematičkim problemima koristeći se nizom strategija koje odražavaju njihove individualne stilove učenja i uključivanja u matematičke situacije. Krutetskii je utvrdio da takve sklonosti tvore matematički sklop uma te je razlikovao tri tipa: analitički, koji misli verbalno-logički; geometrijski, koji misli vizualno-slikovno; i harmonijski, koji objedinjuje oba. Sharma tu distinkciju iznova formulira za potrebe nastavne prakse pod nazivom matematička osobnost, suprotstavljajući kvantitativni tip kvalitativnom uz napomenu da većina pojedinaca pokazuje kombinirani tip. Kvantitativno usmjereni studenti zadatke rješavaju metodično, s naglaskom na jasnoći postupka i preciznosti u primjeni standardnih algoritama, ali mogu imati poteškoća s nerutinskim problemima. Kvalitativno usmjereni studenti problemskim situacijama pristupaju cjelovito, oslanjajući se na intuiciju, vizualne predodžbe i kreativno prepoznavanje odnosa među matematičkim pojmovima, dok im strogo formalizirani postupci mogu predstavljati poteškoću. Ovaj rad ispituje kako se te usmjerenosti očituju u strategijama koje učenici i studenti primjenjuju pri rješavanju problemskih zadataka. Prikupljena su pisana rješenja dvaju nerutinskih problema od 184 studenta četvrte godine učiteljskog studija kroz pet uzastopnih akademskih godina te od 26 učenika četvrtih razreda s posebnim interesom za matematiku, a svako je rješenje razvrstano kao kvantitativno, kvalitativno, kombinirano ili samo odgovor. Cilj je kvalitativnom analizom rješenja utvrditi koja usmjerenost prevladava te ispitati kako se ona odnosi prema strukturi zadatka. Nalazi mogu pomoći učiteljima u prilagodbi nastavnih metoda, u poticanju dubljeg matematičkog mišljenja te u vrednovanju učeničkih radova na način koji uzima u obzir usmjerenost koju učenik unosi u rješavanje.

Ključne riječi: matematička osobnost, problemski zadaci, strategije rješavanja problema, kvantitativno i kvalitativno mišljenje, vizualizacija

Geometry Meets Statistics: Simpson's Paradox

Attila Bölcskei

Department of Building Informatics, Geodesy and Mathematics,
Ybl Miklós Faculty of Architecture and Civil Engineering, Óbuda University,
Hungary

Abstract. In an earlier presentation in Osijek, we showed that the National Curriculum, which was introduced in 2020, contains numerous new elements for the teaching of statistics. One of these new optional contents is Simpson's paradox, which, roughly speaking defines it as a phenomenon that arises when averages, or percentages, are taken across different groups, and these group averages appear to contradict the overall averages.

This phenomenon is one of those statistical problems that can be represented by geometry, making it easier to understand and to remember.

In this paper, we present a possible structure for teaching Simpson's paradox in secondary schools, which relates statistical knowledge to vectors, semi-planes, intervals, set operations, interactive geometry applications, geometric probabilities and mixing problems.

Keywords: statistics education, Simpson's paradox, geometry applications, GeoGebra, curriculum development

1. Introduction

In recent years, there has been a significant shift in focus in Hungarian secondary school mathematics education. As we have already reported in a previous publication and lecture (Bölcskei, 2024), geometry, which has traditionally been considered important, and spatial geometry in particular, have been pushed into the background and replaced by mathematical statistics and probability theory, which now occupy a larger number of lessons. The increased emphasis on statistics, which is closer to real life, is understandable. At the same time, the loss of the spatial and geometric thinking developed by geometry would be a great pity from the point of view of the education of future generations. Previously (Bölcskei,

2024), we made a proposal on how certain data visualization techniques could be used to develop spatial abilities. In this paper, we would like to show how the teaching of the Simpson paradox, which has recently appeared in the Hungarian secondary school curriculum, can be linked to geometric knowledge.

2. Geometry versus statistics

Benoit B. Mandelbrot, the famous mathematician, who can be considered the father of fractals, said in an interview (Staar (1990)) that even as a student he solved algebraic or analytical problems by creating a kind of mental image, a unique, abstract geometry. In the process of solving the problem, he strove to make the often confusing, asymmetrical shapes "more beautiful" and "more symmetrical", which often led to novel solutions. At the Ecole Normal, where he studied at the time, he was able to solve abstract analytical problems by geometrizing them.

Our goal, although much more modest, is similar to Mandelbrot's. In the science of statistics, which, to put it bluntly, involves the examination of boring sets of numbers, we want to find the points of connection where the problems under investigation can be made more visible. We firmly believe that reformulation and geometric interpretation open up new possibilities for solving problems and help deeper understanding. Geometry allows imagination and even intuition to be brought into the solution of problems.

Our task is not easy, as the approaches, thinking, and tools of statistics and geometry are completely different. The subject of statistics is much more abstract than that of elementary geometry, which deals with points, plane figures, simple solids, and their transformations. Statistical data sets and the correlations that can be found in them are much more difficult to grasp. Proof plays a classically important role in the tools of geometry, guaranteeing a statement to be 100% true if certain conditions are met. Statistics, on the other hand, favours empirical methods, and its statements are often only valid at a certain level of significance. Statistics works with estimates rather than precise measurements. The difference between geometry and statistics is well illustrated by the fact that their applicability also differs greatly. The former is used in physics and applied sciences (e.g., engineering), computer visualization, and robotics. The latter was first discovered by economics and the social sciences. In geometry, intuition and "vision" or visuality, play an enormous role, while the handling of uncertainty is of very little importance. The opposite is obviously true in statistics. Geometry has both a speculative, theoretical side and practical applications. Statistics, on the other hand, is more practical and aims to produce tangible results. All these statements are, of course, rather superficial, as statistical theories were also born out of serious abstraction. However, they illustrate well the mathematical understanding at the high school level.

After all this, it is appropriate to list the areas where statistics and geometry intersect. This is primarily data visualization, where we characterize populations with shapes or even figures (Bölcskei (2023), Bölcskei (2024) and Bölcskei et al., (2024)) and draw conclusions from them. Another area where probability

theory (which is not separated from statistics in Hungarian public education) and geometry work together is the calculation of geometric probabilities. In this case, we illustrate our outcome space with, for example, a square, and the individual outcomes appear as planar figures.

In this paper, we would like to present a new connection where the concept of vectors in geometry helps us understand a statistical anomaly, the so-called Simpson's paradox.

3. About the Simpson's paradox

Descriptive statistics textbooks often discuss the phenomenon known as the Simpson's paradox, which we will now briefly describe based on Anderson et al. (1999). Data in two or more crosstabulations are often aggregated to produce a summary crosstabulation showing how two variables are related. In such cases, conclusions, drawn from the original crosstabulations can be reversed when the data are aggregated into one. The reversal of conclusions based on aggregate and unaggregated data is called Simpson's paradox.

It is often cited as an example of why it is important to teach statistics.

In Hungary, the 2020 Framework Curriculum prescribes that the final chapter of descriptive statistics should be taught in 12 lessons in grades 11–12. As part of this, the recommended activities include presenting Simpson's paradox through examples. The obvious reason for this may be to show students that simply presenting data can often be misleading.

Simpson's paradox, on the other hand, states that decision-making is not possible because the conclusions drawn from probabilities at different levels contradict each other. Without Simpson's paradox, decision-making could be carried out at an aggregate level because examining finer details would either become unnecessary or would not be contradictory (Hovhannisyan & Allahverdyan (2024)).

We note that Simpson's paradox is interpreted more broadly in the literature. Simpson's paradox also includes cases where the association between X and Y in the entire group disappears in each sub-group, as well as cases where an association between X and Y appears in each sub-group even if they are unassociated in the whole group (Dong et al. (2024)).

We also mention that our paradox is in close relation with the so-called amalgamation paradox, which is roughly speaking a statistical phenomenon in which the marginal or unconditional association of two variables, X and Y, falls outside the range of the conditional associations with respect to a third variable Z (Dong et al. (2024)). In fact, the Simpson's paradox is an extreme, special case of the amalgamation paradox. While in the latter case the only condition is that the aggregated total measure falls outside the range of the subgroup measures, the Simpson's paradox involves a complete reversal of the inequality. According to the geometric approach presented later, this phenomenon of amalgamation – in the broader sense – occurs because the slope of the resulting sum vector is not strictly bounded by

the slopes of the component vectors, as it is strongly influenced by the magnitudes of the individual vectors representing the varying sample sizes.

It should also be noted that some studies attribute the Simpson's paradox to a third variable. That is, Simpson's paradox is an obstacle to establishing a probabilistic association between two events a_1 and a_2, given the third (lurking) random variable B. It may occur that the random variables A (which combines a_1 and a_2, and their complements) and B have a common cause C that need not be observed (Hovhannisyan & Allahverdyan (2024)).

For more information on the history of Simpson's paradox, we recommend the paper Pearl (2013).

4. An example and its geometric interpretation

Consider the table below and the related task.

The table shows the annual wage bill and number of employees of two companies (A and B), broken down by gender and in total. Decide which company is worth working for based on the expected salary.

Table 1. Data table for the example.

	Company A		Company B	
	Annual wage bill (hundred thousand euros)	Number of employees	Annual wage bill (hundred thousand euros)	Number of employees
Men	24	50	10	20
Women	3	10	10	30
Total	27	60	20	50

Due to its simplicity, this task could actually be set in 8th grade, as it only requires knowledge of how to calculate averages.

The solution clearly shows that Company A performs better overall, as the total annual salary here is $\frac{2700000}{60} = 45000$ €, while at Company B it is only $\frac{2000000}{50} = 40000$ € per year.

On the other hand, if I am a man looking for a job, it is worth going to Company B, as $\frac{1000000}{20} = 50000 > \frac{2400000}{50} = 48000$. However, if I am a woman looking for a job, then Company B is again the better choice, as $\frac{1000000}{30} \approx 33333$ € $> \frac{300000}{10} = 30000$ €.

In short, the averages are smaller at Company A, yet the main average is higher here.

This example explains what we mean when we say that conclusions are reversed if we look at decomposed crosstabulations rather than the aggregated crosstabulation.

It may be interesting to ask students to explain the reason for this paradox. It is conceivable that many will realize that the difference is caused by the different structures of the two companies, in other words, if the distribution of sample sizes is unfavourably skewed. This explanation can obviously only be expected from students in grades 11–12, and even then, not from everyone.

As an introductory task, let us ask the range the average earnings at each company might fall within. What determines the value the main average will take?

In the answer (for Company A [30,000; 48000], Company B [33333; 50000]), we must point out that the two sets are not disjoint. The fact that in case B, each of the endpoints is greater than the endpoints of the interval in case A does not mean that this is true for every element of the sets.

The value of the main average takes must lead to the concept of weighted average. This describes the ratio of men to women working at each workplace. This brings students closer to understanding that any intermediate value can be achieved by varying the ratios appropriately, which is the core of the explanation of the Simpson's paradox.

For example, pose the task of finding numbers (ratios) for which the average salary in Company A would be €42,000 and for which the average salary in Company B would be only €36,667 (rounded).

In solving the problem, all students should realize that the task can be solved in all cases when the desired average is within the interval.

The question now is the following: supposed that in the task the Simpson's paradox holds, what values can the average earnings range between? Or, conversely, for which pairs of values does the Simpson's paradox not hold and plausible reasoning apply?

To answer this, set operations must be performed on the intervals, while paying attention to the endpoints. The Simpson's paradox definitely does not hold if the average earnings at Company A fall within the interval [30000; 33333] or if those at Company B fall within the interval [48000;50000]. There is a chance that Simpson's paradox will occur if the average earnings fall within the open interval]33,333; 48,000[.

Here we draw attention to a possible analogy. Simpson's paradox seems complicated and surprising in its formulation. At the same time, its reformulation as a mixing task can hardly be called surprising. Let us assume that there are two bottles of solution on a table, containing, say, 10% and 50% alcohol. On another table, there are two bottles containing 20% and 55% alcohol. In both cases, we will mix the contents of the two bottles in some proportion. Is it true that the mixture produced on the first table will always have a lower alcohol content than the one produced on the second table? Due to its practicality, students can easily answer this question.

For graphical representation, i.e., to illustrate the Simpson's paradox using planar vectors, we first formally write down the problem with the students, using letters and inequalities.

One form of this could be as follows: $\frac{a_1+b_1}{a_2+b_2} > \frac{c_1+d_1}{c_2+d_2}$, whilst $\frac{a_1}{a_2} < \frac{c_1}{c_2}$ and $\frac{b_1}{b_2} < \frac{d_1}{d_2}$ simultaneously hold.

In the table, the numbers in the second column (number of employees) are given an index of 1, while the values in the first column (total wages) are given an index of 2.

Let us now assign the corresponding pairs of values written in this way to pairs of numbers, and those to points on the plane, and then consider their position vectors. The sum is then geometrically the sum vector of the vectors. The ratio of the coordinates of the vectors represents the slopes of the individual radius vectors, and the length of the vector is irrelevant. An illustration shows that the two red vectors representing men and women working at Company B are steeper than the two black vectors representing the same at Company A, but the sum of the red vectors is less steep than that of the black vectors.



Figure 1. Illustration of the original problem (source: own edition).

We would like to further demystify Simpson's paradox by teaching students how they can create similar tasks themselves. To do this, we use the interactive mathematics software GeoGebra (www.geogebra.org). We have previously used GeoGebra ourselves to simulate data visualization in a completely different context (Bölcskei (2023), Bölcskei et al., (2024) and Talata et al. (2023)).

To achieve our goal, we first take two arbitrary "black" vectors and one red vector in such a way that the slope of the black vectors is always smaller than that of the red one. To be short, our single red vector is steeper than either of the two black ones. We calculate the sum of the two black vectors. Then, taking advantage of the possibilities offered by dynamic geometry software, let us take the second vector, whose endpoint we will move. On the one hand, the goal is for the red vector to be steeper than the other black one, yet the sum of the two red vectors to be less steep than the black ones. First, we can let the students experiment to

find the appropriate endpoint and thus the second vector – a fairly easy task. Then try to discover the range by moving the point, where the task they have created describes the Simpson's paradox. It is likely that several of the students will find the V-shaped range, as the solution.

In the next step, we write down our specific task algebraically and using GeoGebra demonstrate the range where the endpoint of the second vector can be.

This domain will obviously be the intersection of two half-planes. The first half-plane consists of the points on the plane that lie above the line defined by the unpaired black vector with smaller slope. The other half-plane is obtained as follows. The inequality $\frac{a_1+b_1}{a_2+b_2} > \frac{c_1+d_1}{c_2+d_2}$ should hold, where a_i, b_i *and* c_i are known constants. Rearranging this and considering that all numbers are positive, we get that $\frac{(a_1+b_1)(c_2+d_2)}{a_2+b_2} - c_1 > d_1$. That is, we have a linear relationship between the two variables d_1, d_2 i.e., we have obtained a half-plane bounded by a straight line, which (due to the direction of the inequality) lies below. The following figure illustrates the reasoning in a specific case.

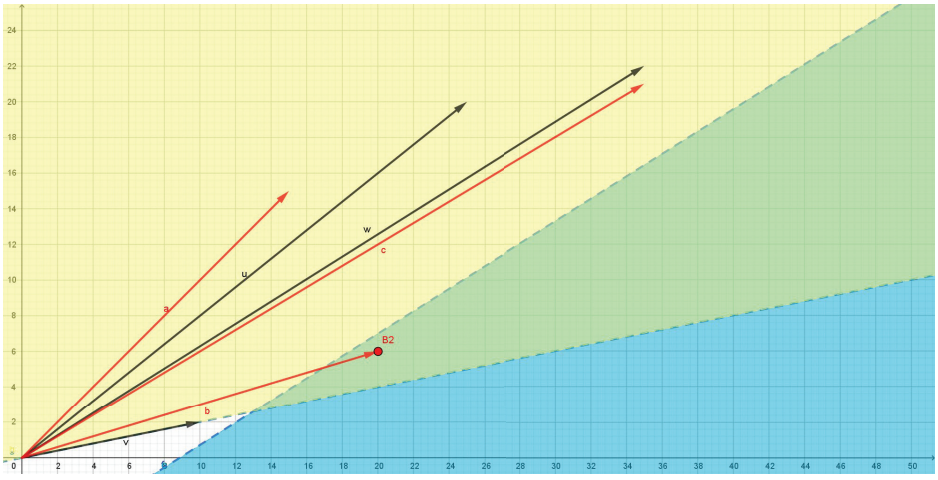

Figure 2. Illustration of how to find the second vector (source: own edition).

For any point chosen in the common part of the two half-planes, if added to our data table, we arrive at Simpson's paradox.

We may ask whether it is sure that the two half-planes will always have a common part in the first plane quarter. To answer this, we need to note that the slope of the line bounding the second half-plane is $\frac{a_1+b_1}{a_2+b_2}$, which is equal to the slope of the black sum vector and is therefore always steeper than our less steep black vector.

Further lines of thinking can be developed in relation to the task. For example, we can look for an answer to the question of whether we can generalize the introductory task of Simpson's paradox to multiple companies. In this case, we require a relationship opposite to the main average for the averages of let's say

three companies. With GeoGebra, we can easily answer this question. Without a deeper examination of this (and the case of n companies), we will simply illustrate the analogous answer with a figure.

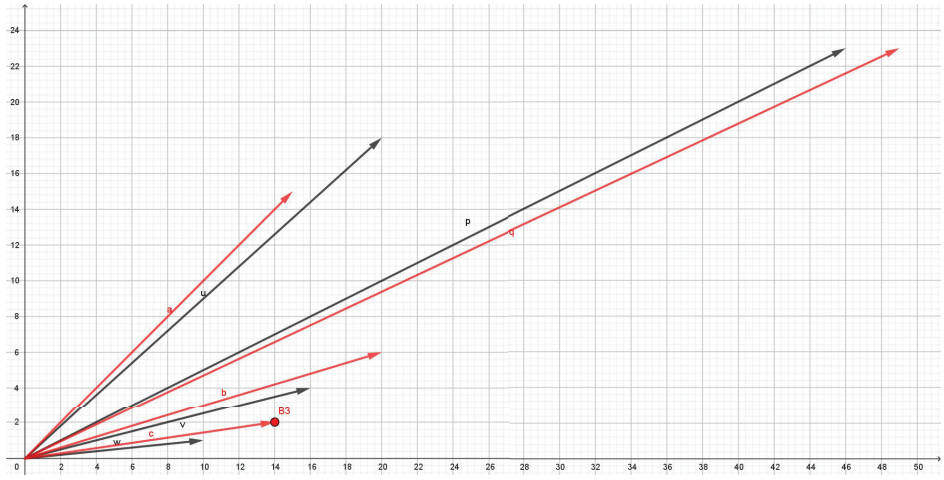

Figure 3. Illustration of the general problem (source: own edition).

The following question can also be raised. Given a half-finished data table where only two related data (representing one vector) are missing. If we write in arbitrary positive numbers, what is the probability that the resulting data table will exhibit Simpson's paradox? The problem can obviously be solved using geometric probability, where the question is the probability of falling into the V-shaped range. We leave it to the Reader and students to answer this question. (Hint: use angles as parameters for the whole sample space and for the V-shaped region.)

5. Summary

The Simpson's paradox is a popular topic in descriptive statistics, drawing attention to the limits of plausible reasoning. In our paper, we have tried to draw attention to the fact that this well-known paradox does not seem paradoxical at all to those who have mastered mathematical thinking. We cannot emphasize enough the importance of teaching mathematics in a way that it truly teaches thinking, rather than just training children to solve problems in a prescribed way. The above examination of the paradox nicely illustrates how everyday problems can and indeed should be understood and solved with a mind trained in mathematics.

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Contact addresses:

Attila Bölcskei
Department of Building Informatics, Geodesy and Mathematics
Ybl Miklós Faculty of Architecture and Civil Engineering
Óbuda University
Thököly út 74, 1146 Budapest, Hungary
e-mail: bolcskei.attila@uni-obuda.hu

Ahol a geometria és a statisztika találkozik: a Simpson paradoxon

Bölcskei Attila

Építésinformatikai, Geodéziai és Matematikai Tanszék
Óbudai Egyetem Ybl Miklós Építéstudományi Kar, Magyarország

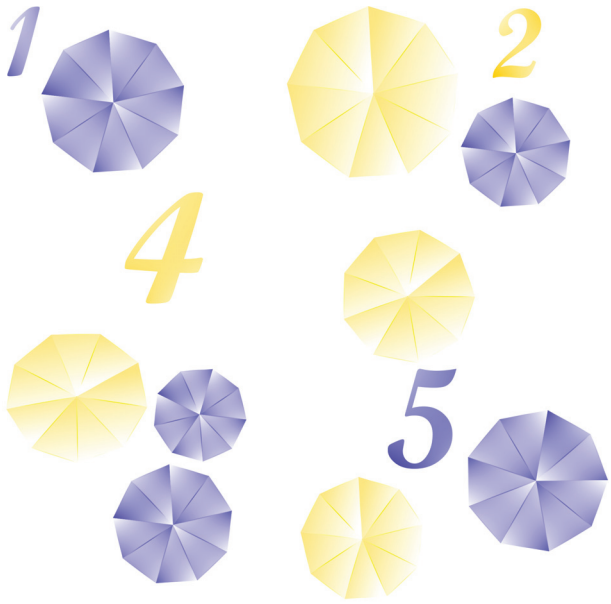
Absztrakt. Egy korábbi, eszéki előadásomban azt mutattam be, hogy a 2020-ban bevezetett Nemzeti Alaptanterv milyen új elemeket tartalmaz a statisztika oktatásának vonatkozásában. Egy ilyen új, opcionális tananyag a Simpson paradoxon is, mely pongyolán fogalmazva arról szól, hogy amikor különböző csoportok átlagait vagy százalékos arányait vesszük, előfordulhat, hogy ezek a csoportátlagok ellentmondanak a főátlagoknak.

A jelenség azon statisztikai problémák közé tartozik, mely a geometria eszközeivel szemléltethető és lényege ezáltal egyrészt könnyebben megérthető, másfelől maga a probléma könnyebben megjegyezhető.

Az előadásban bemutatjuk a Simpson paradoxon középiskolai tanításának egy lehetséges felépítését, mely a statisztikai ismereteket a vektorokkal, félsíkokkal, intervallumokkal, halmaz műveletekkel, dinamikus geometriai szoftverekkel, geometriai valószínűségekkel és keverési feladatokkal is kapcsolatba hozza.

Kulcsszavak: statisztika oktatása, Simpson paradoxon, geometria alkalmazása, Geogebra, tananyag fejlesztés

2.
Innovative Teaching Approaches, Representations, and Inclusion in Mathematics Education



Visualizing Division in Mathematical Textbooks

Jasmina Milinković[1]* and Nives Baranović[2]

[1]Faculty of Education, University of Belgrade, Serbia
[2]Faculty of Humanities and Social Sciences, University of Split, Croatia

Abstract. Mathematics textbooks are fundamental resources for both teachers and students, shaping the way mathematical concepts are introduced and understood. This study explores how the concept of division has been visually presented in primary-grade textbooks across different historical periods. Specifically, it examines the types of visual representations used in introductory division lessons in Serbia and Croatia from the 1950s to the present. The analysis is grounded in Vergnaud's theory of multiplicative structures and Greer's classification of multiplicative situations, providing a framework for evaluating how different representations support conceptual development. By comparing historical and contemporary approaches, the study seeks to identify trends, explain differences, and assess the role of visualization in fostering a deeper understanding of division. The findings contribute to the discussion on how visual representations can effectively support students' learning and have implications for textbook design and didactical approaches in mathematics education.

Keywords: arithmetic operation, types of division, teaching resources, visual mediators

1. Introduction

Learning mathematics is a gradual process, through which meaning is built up by comparing selected problem situations and utilising diverse systems of representations and tools, since mathematical objects are not directly accessible to the senses (Bruner, 1960; Sfard, 2008). Contemporary society transmits information primarily through visual forms, as technologies foster and reinforce communication that is fundamentally visual (Arcavi, 2003). Arcavi highlights three functions of visual representations:

*Corresponding author.

1) as a support and illustration of symbolic representations;
2) as a tool for resolving conflict between intuition and symbolic solution and
3) as a tool to reengage and recover conceptual understanding (discussed in Milinković, 2024).

In mathematics, visualization is understood as the process of creating and transforming visual mental imagery, as well as interpreting spatial inscriptions that play an important role in mathematical activity (Presmeg, 2006). Mathematics, as a discourse, includes four specific characteristics: word use, visual mediators, endorsed narratives, and rituals (Sfard, 2008, p. 133). Visual mediators, such as symbolic notations, tables, graphs, representations and didactic tools, shape concept formation by directing cognitive processes and communication (Baranović & Milinković, 2025).

Mathematical thinking and reasoning develop the ability to identify patterns across diverse contexts and varying levels of abstraction, and to express these patterns in mathematical form. Effective mathematical thinking further requires an understanding of how different representations convey the same concept, as well as an awareness of the structural similarities and distinctions among representational systems (Goldin, 2008). A central question for this study is: How does visualization in textbooks contribute to the formation of the concept of division as an arithmetic operation? To explore this issue, we examine a selection of mathematical textbooks from past and present times.

Mathematical textbooks have long served as primary instructional resources, structuring the way division is presented and shaping students' conceptual understanding. By analysing historical and modern perspectives taken in textbooks, we assess how textbook approaches compare with current theoretical perspectives. This study focuses on the visual representations of division in primary education, specifically as proposed in a textbooks published in former Yugoslavia and present-day Croatia and Serbia. The analysis is informed by theoretical perspectives on multiplicative reasoning, allowing us to interpret how visual mediators align with different conceptualizations of division. Through this examination, the study contributes to a broader understanding of how visual representations in textbooks can support or constrain the development of mathematical concepts, and how the design of visual mediators reflects prevailing didactical orientations over time.

2. Theoretical background

Arithmetic operations are illuminated in different theoretical frameworks. The development of multiplicative reasoning emerges as children move beyond repeated addition to understanding division as the inverse process, enabling them to interpret and solve problems involving equal groups and sharing (Anghileri & Johnson, 1988). In this paper we briefly introduce part of the theories of Vergnaud, Greer and Zweng, taking into consideration only the parts that are relevant for the analysis of arithmetic operation of division in primary grade textbooks.

Gérard Vergnaud's (1988) framework of multiplicative structures situates multiplication and division within a broad conceptual field encompassing "all situations that can be analyzed as simple and multiple proportion problems and for which one usually needs to multiply or divide" (p. 141). Within this field, he distinguishes three major classes of problems: isomorphism of measures, product of measures, and multiple proportions.

The first class, *isomorphism of measures*, involves a direct proportion between two measure spaces, independent of the numerical domain. Vergnaud identifies seven canonical problem types within this class: equal groups, equal measures, rate, measure conversion, multiplicative comparison, part–whole relationships, and multiplicative change (Vergnaud, 1988). Because the other two classes, *product of measures* and *multiple proportions*, are more pertinent to post-primary mathematics, they are not addressed in this article but are discussed elsewhere (Milinković, 2022).

Recent research underscores the continuing relevance of Vergnaud's framework for understanding how children develop multiplicative reasoning. For example, Lamon (2023) highlights that coordination of multiplicative comparisons remains a persistent challenge even when students can perform computations, suggesting the need for instruction that explicitly links equal-groups problems to ratio and rate reasoning. Similarly, Harel and Confrey (2022) argue that early experiences with both measurement and partition division foster a more flexible understanding of rational number concepts, which are foundational for proportional reasoning in later grades.

Division, as a key component of multiplicative reasoning, is commonly interpreted through two complementary perspectives: measurement division and partition division (Zweng, 1972). *Measurement division* asks how many subsets of a given size can be formed from a larger set. For example, determining how many groups of three can be made from twelve. The "sets" in question may be discrete (numbers, apples, candies) or continuous measures such as length, area, or volume. *Partition division*, by contrast, begins with a fixed total and requires distributing it into a specified number of equal subsets, such as sharing twelve cookies equally among four friends. These two perspectives, measurement and partition, shape both the design of division tasks and the ways students come to understand division in the classroom.

2.1. Greer's classification of multiplicative situations

Greer's (1992) classification foregrounds the diverse semantic structures that underlie multiplication and division problems. By moving beyond rote computation, it illustrates how the context of a problem shapes the meaning of the operation and the strategies students employ.

A growing body of classroom-based research underscores the instructional value of these categories. For instance, Greer and colleagues (2024) show that number-line and area-representations can help students bridge discrete (set-based)

and continuous (measure-based) contexts, reinforcing Vergnaud's (1988) claim that multiplicative structures extend beyond whole numbers. Such findings suggest that tasks employing multiple representations, length or area for measurement division, and sharing scenarios for partition division, strengthen learners' ability to generalize across contexts.

Table 1. Greer's four types of multiplicative situations (Greer, 1992).

Type	Description	Example
Equal groups	Repeated groups of the same size	4 bags with 5 apples each $\rightarrow$ $4 \times 5 = 20$ apples
Multiplicative comparison	One quantity is a multiple of another	Ana has 3 times as many books as Tom $\rightarrow$ $3\times$Tom's books
Rate (Proportionality)	Relationship between two measures with a constant ratio	\$2 per kg of apples $\rightarrow$ 3kg costs $3 \times 2 = \$6$
Combinations	Number of ways of pairing items from two sets	3 shirts and 4 pants $\rightarrow$ $3 \times 4 = 12$ outfit combinations

Greer's framework is particularly useful for identifying the cognitive demands of different problem types and for supporting the development of flexible, context-sensitive strategies. It aligns with Vergnaud's conceptual field of multiplicative structures by highlighting that multiplicative reasoning is multifaceted and emerges progressively as students engage with varied situations.

Division, while the inverse of multiplication, requires more than procedural fluency. Learners must distinguish between partition division (dividing a set into a specified number of equal subsets) and measurement division (determining how many subsets of a given size can be formed), recognizing the underlying multiplicative relationships. Greer's classification enriches this perspective by revealing the semantic diversity of division problems and the need for flexible reasoning across them.

An important issue is how learners conceptualize the link between multiplication and division. McCrink et al. (2007) observed that students do not consistently exploit the inverse relationship of multiplication and division as a shortcut when problem-solving. Milinković (2025) asserts that these findings underscore the need for instructional approaches that foster a deeper, more flexible understanding of the interdependence between multiplication and division.

Taken together, the perspectives of Greer and Vergnaud provide a robust theoretical foundation for analyzing how textbooks present division and for evaluating how different visual representations support or constrain children's development of multiplicative reasoning.

2.2. Division in primary grades curriculum

In didactics of primary mathematics, several focus areas emerge: (1) the concept of equal sharing (partition/measurement division); (2) the relationship between

division and multiplication; (3) the relationship between division and subtraction; (4) understanding the difference between partitioning and measurement division; and (5) division with remainders.

Division is typically introduced as the process of either sharing a quantity into a given number of equal parts or determining how many times a given quantity can be contained within another quantity. In Serbian curriculum (ZUOV, 2021) division is introduced by analogy to multiplication using the multiplicative scheme. To build the concept of division pupils analyze problems in real context that can be described as the decomposition of sets with a finite number of elements into disjoint subsets with equal number of elements (Milinković, 2025).

Approaches to conceptualizing division are designed to connect with students' prior experiences of other learned arithmetic operations, making the new concept accessible and meaningful. Hence, division is often presented as the inverse of multiplication and/or as repeated subtraction. Establishing these connections between operations is crucial for building fluency and reinforcing conceptual understanding. When taught as repeated subtraction, division supports the development of the idea of sharing equal quantity through a concrete procedural model situation. For example, pupils could be asked to consider situation of sharing six apples among two kids. The process of division by subtraction would be modelled: first, one apple would be given to each child, then, the process would be repeated until all six apples are handed out. Understanding division involves not only procedural fluency but also the development of structural insight into multiplicative relationships with consideration to the complexity and diversity of contexts in which division operates. The problem types correspond to different cognitive schemes and invariants.

3. Methodology

Every primary mathematics textbook relies on visual mediators to support the introduction of new concepts. Common representations of division include sets, number lines, arrays, and bars, all of which aim to make the operation more concrete and to strengthen conceptual understanding. However, it remains an open question: which visual tools are considered most appropriate for introducing the concept of division in its earliest stages. This study therefore focuses on identifying and analysing the types of visual representations used when division is first introduced in textbooks.

3.1. Research Questions

RQ1: How has the concept of division been visually introduced in primary-grade textbooks?

RQ2: Has there been a consistent approach across time, or can we observe changes in the preferred visual mediators?

3.2. Sample

The study is based on a convenience sample of 21 textbooks published in former Yugoslavia, Croatia, and Serbia between 1957 and the present. We listed them in the chronological order:

1. Filipović, M. & Koletić, S. (1957). *Računamo i mjerimo*. Školska knjiga.
2. Prvanović, S. (1963). *Matematika 1*. ZUNS.
3. Bajić, S. (1971). *Matematika 2*. ZUNS.
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18. Boras Mandić, A., Lončar, L., Pešut, R. & Križman Roškar, M. (2020). *Nina i Tino 2*. Klett.
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21. Ivančević Ilić, I. & Tahirović, S. (2021). *Matematika za 2. razred*. Logos.

We did not separate textbooks used in Croatia from those used in Serbia, because there was a significant overlap in the period up to the 1990s. So, the focus was not on comparison but on identification of historic trends, explanation of differences, and assessment of the role of visualization in fostering a deeper understanding of division.

3.3. Analytical framework

The theoretical frameworks provided interpretive lenses for understanding how representations map onto problem types. The triangulation enabled a systematic comparison of historical and contemporary practices. This methodology enabled us to trace both continuity and change in the ways textbooks introduce division, offering insights into how visual tools reflect broader curricular and didactical orientations.

For each textbook, the first lesson on division was examined. Each lesson, regardless of the time the textbook was published, consists of one or few problem situations serving as a prompt for introduction of a new concept. The unit of analysis were those problem situations. In the analysis we reflected on dimensions of the problem situation: type of multiplicative situation, visual representation, connections to other operations, and context. A qualitative analysis started from the classifications proposed by Zweng (1972), Greer (1992), and Vergnaud (1988) but limited to the elements of classification relevant for 2nd grade school level. For example, coding scheme (Table 2) recognizes two distinct mathematical structures of the problem situation as "partition" vs. "quotition" (Greer, 1992). We also identified the type of visual representation of the problem as: array, set, number line, realistic picture, bars (Bruner, 1960). Table 2 presents whole coding scheme.

Table 2. Coding scheme for analysis.

Type of multiplicative situation (Greer)	**Type of division (Zweng)**	**Visual representation (mediators)**	**Context**	**Connection between division and other arithmetic operations**
Equal groups /partitioning	partition division	Set	Real context	division – subtraction
Multiplicative comparison /quotition	measurement division	Array	Mathematical context	division – multiplication
		Number line		
		Bar		

Two researchers coded independently, and disagreements were discussed until a consensus was reached.

4. Results

4.1. Before New Math

In the period between 1945 and the 1970s, division used to be introduced in the 1st grade. In a selected textbook from that period, division was introduced as the inverse operation to multiplication. With a loosely structured curriculum, division was not introduced in a separate sequence of lessons. The first example in which the concept of division was visualized using array representation is presented in Figure 1.

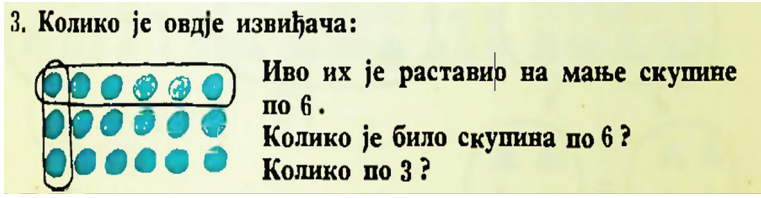


Figure 1. Array visual representation of division.

Transcript of Figure 1:
3. How many scouts are there? Ivo separated them into smaller groups of 6. How many groups of 6 were there? How many of 3?

Context of this problem is dividing group of scouts into smaller equal groups of 6 or 3. Pupils are asked to investigate the problem situation. This is an "equal group" structure of the problem (Greer, 1992). This case of problem requires the process of determining how many times number 6 (or number 3) is included in 18 (Vergnaud, 1988). The first column and the first row are circled, helping students to understand the situation and suggesting possible strategy for finding the answer. Although the context is realistic, authors opted for circles to represent people, each circle represents one scout. The group of scouts is set to be in array arrangement, to ease investigation. This math textbook provides no word explanations with drawings. The array representation with circles was dominant in the succeeding math problems, regardless of the context of the problems.

In another textbook from that period (Filipović & Koletić, 1957, p. 34), a first encounter with division appears before subtraction or multiplication are introduced (Milinković, 2025). The illustration shows a basket of 18 beans and another of 6, separated by a three-dot symbol – a clear instance of measurement division (Figure 2). A nearby image with 13 beans and a basket of 2 depicts division with a remainder. Without text, these visuals invite students to observe whether quantities can be evenly distributed, encouraging intuitive reasoning and discovery through pattern recognition.

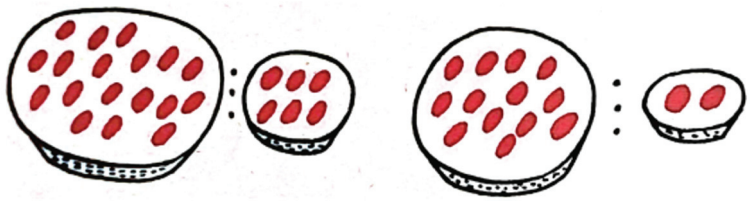


Figure 2. Measurement division with and without remainders.

Down the page of that 1st grade textbook (Filipović & Koletić, 1957, p. 34), array representation was used in a visually posed sequence of problems involving determining possible groupings and divisors (Figure 3). These problems were used for early exploration of divisibility using only visual cues.

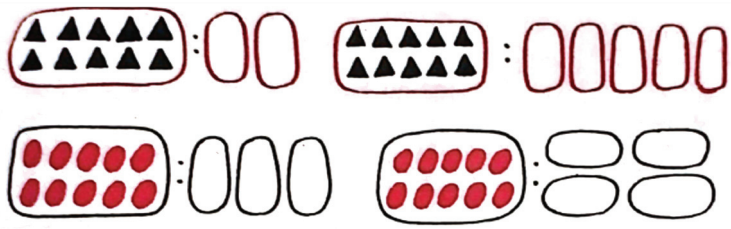


Figure 3. Determining the divisor in a partition division task.

4.2. New Math period

In the next decade, the New Math approach became dominant. In primary grades, it meant extensive relying on the Set theory and Logic. Markovac and Benčić's textbook (1977, p. 82) has an illustrative example with an equal grouping problem situation presented with a set representation shown in Figure 4. The task of division is solved by subtraction. The authors explain to readers: "It works like this: Take 5 apples from set S and put one on each plate. Then take 5 apples again and put... Continue until divided all, 3 apples each. In numbers, we write this operation like this: $15 : 5 = 3$". Thus, it is assumed that pupils are very familiar with the term *set*.

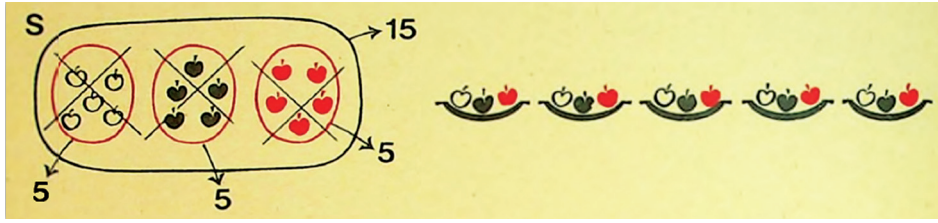


Figure 4. Dividing apples in groups of three apples and among three kids.

Here is a set representation visualizing the connection between multiplication and division (Figure 5) in a lesson titled "Multiplying by 2 and dividing by 2 (double, half)" (Markovac and Benčić, 1977, p. 84).

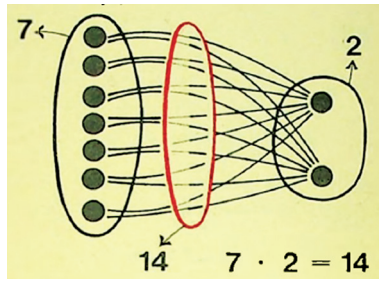


Figure 5. Representation of the connection between multiplication and division.

Measurement division and partitive division can be found within the same problem (Zweng, 1972). Pupils are asked to "imagine 12 candies" and to separate them into groups of 3 candies. The question is how many groups of candies they would create in that way. After this "measurement division" problem, further down the line is a "partitive division" case situated within the same context, "how many candies would get a child if three of them are sharing those 12 candies" (Figure 6). When analysing the later problem, pupils are directed to notice relation between division and multiplication. It is stated that "There are 4 times 3 in 12 or 3 goes into 12 four times because 4 times 3 is 12 and 12 divided by 3 is 4 (because 3 times 4 is 12)".

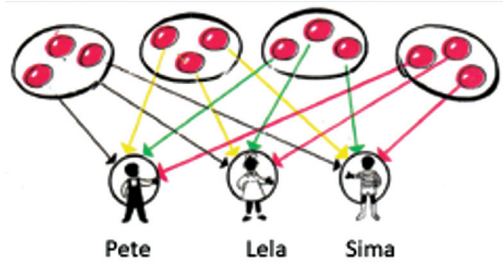


Figure 6. Dividing candies into groups of three and among three kids.

Division is linked with subtraction in a 1970s textbook (Figure 7).

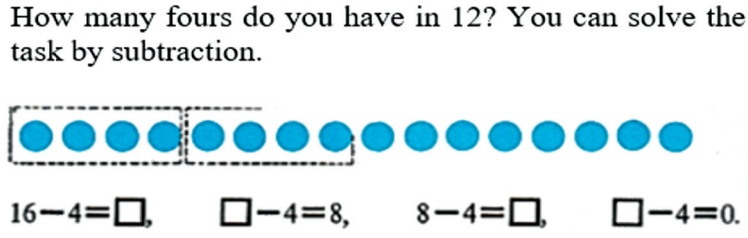


Figure 7. Division as a repeated subtraction in a 1970s textbook.

In the next decade, still division is introduced as a repeated subtraction but for visualization two new mediators are used – a number line and a bar as forms of visual representation (Figure 8, Figure 9).

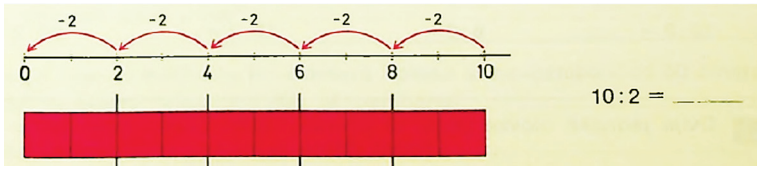


Figure 8. Number line and bar.

As a creative way of visualization explanations are illustrated as a cartoon with number line as a mediator for a partitioning problem situation for equal sharing (Đurović & Đurović, 1986, p. 67).

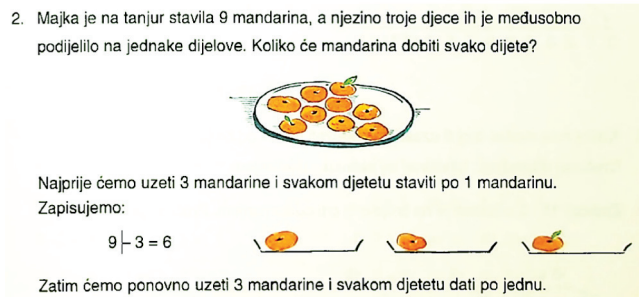


Figure 9. Number line.

Transcript of Figure 9:
1st *frame: When two of us split 10 candies, each of us will have 5 candies.*
2nd *frame: We would write* $10 : 2 = 5$ *and read 10 divided by 2 is 5.*

4.3. Period of constructivism

In the 1990s there were multiple publishers in Croatia, while in Serbia there was still one state publisher. A commonly realistic contextual problem situation of equal sharing was dominantly used as an introductory example (Jagodić, Mrkonjić & Kutnjak, 1998, p. 71; Polonijo & Šakić, 1998, p. 76). Division is introduced as a repeated subtraction (Figure 10a) or alternatively as an inverse to multiplication (Figure 10b). In the first case (Figure 10a), the process of division by repeated subtraction is visually documented in a sequence of pictures showing in the first step how three tangerines are put into three bowls, in the next step, the next three tangerines, and in the third step, the process of partitioning finished as all tangerines were placed in bowls (equal split) (Vergnaud, 1988; Greer, 1992). In the second textbook, as can be seen in Figure 10b, the process of partitioning is not explained. The illustration shows the result of the process of equal sharing of eight cakes between two persons.



a)



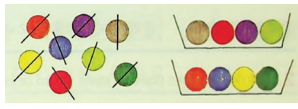
b)

Figure 10. Division as equal sharing.

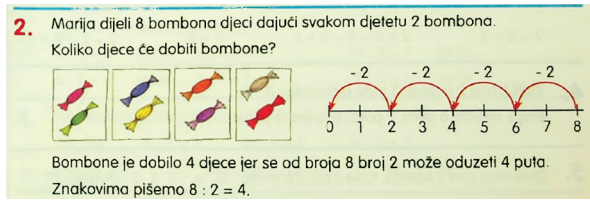
Transcript of Figure 10 a):
Above the 1st image: Mother put 9 tangerines on a plate. Her three kids distributed the tangerines in equal parts. How many tangerines did each child get?
Above the 2nd image: Firstly, we shall take 3 tangerines and give 1 tangerine to each child. We write.
Above the 3rd image: Next, we will again take 3 tangerines and give to each child one (tangerine).
Above the 4th image: The remaining tangerines will also be distributed in the same way, each child will get one tangerine.

Transcript of Figure 10 b):
8 cookies should be distributed equally between two pupils so that each pupil gets the same number of cookies. Everyone will get 4 cookies, because $2 \times 4 = 8$.

Ćurić and Markovac (1997, p. 37) link division with multiplication, yet in the first example, explain division as a repeated subtraction in a situation of equal sharing (Figure 11).



a)



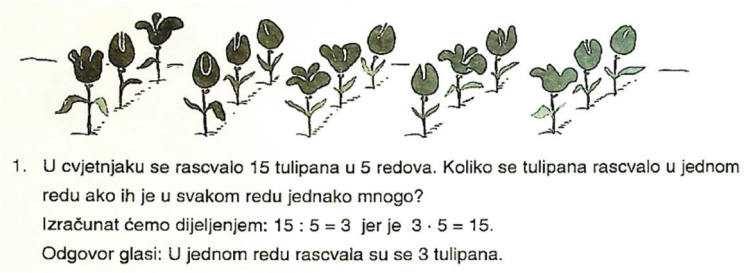
b)

Figure 11. Illustration of the process of division.

Transcript of Figure 11.b:
Above the image: Maria distributes candies to the children, giving each child 2 candies. How many children will get candies?
Below the image: 4 children got the candy because 2 can be subtracted 4 times from 8. We write it in symbols: 8 : 2 = 4.

In the same lesson, in the next situation Maria had to share candies so that each child gets two candies. This is a measurement division task, determining how many subsets of a given size can be formed. The problem is solved visually on a number line via repeated subtraction.

The following decade brought more diversity. After years of omitting array representation, textbook authors used it again as an anchor mediator (Figure 12).



1. U cvjetnjaku se rascvalo 15 tulipana u 5 redova. Koliko se tulipana rascvalo u jednom redu ako ih je u svakom redu jednako mnogo?
   Izračunat ćemo dijeljenjem: 15 : 5 = 3 jer je 3 · 5 = 15.
   Odgovor glasi: U jednom redu rascvala su se 3 tulipana.

Figure 12. An array representation in realistic context.

Transcript of Figure 12:
15 tulips bloomed in the garden in 5 rows. How many tulips bloomed in one row if the same number bloomed in each row?
We will calculate that by division: $15 : 5 = 3$, *because* $3 \times 5 = 15$.
The answer is: 3 tulips bloomed in one row.

Array representation, accompanied with visual explanation, was used in the Serbian textbook as well but with no realistic problem situation (Figure 13). In the same decade, a representation of chocolate appeared in a textbook.

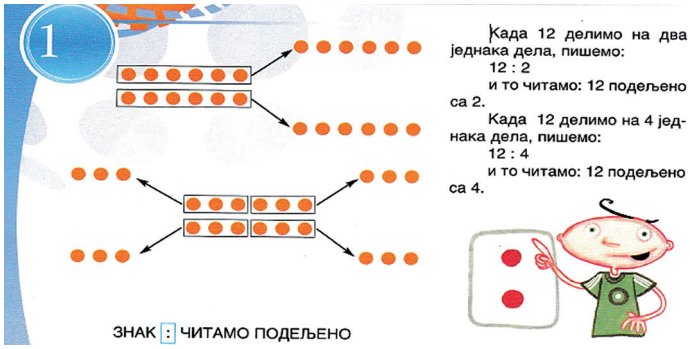


Figure 13. Array iconic representation.

Transcript of Figure 13:
When we divide 12 into two equal parts, we write 12 : 2 , and we read it: 12 divided by 2.
When 12 is divided into 4 equal parts, we write: 12 : 4, and we read: 12 divided by 4.

In the present day, division is introduced as the fourth arithmetic operation, linked to an implicit, unconscious, and primitive intuitive model (Fischbein, 1987). By analogy to introduction of inverse operation of multiplication, the arithmetic operation of division is formed by observing the breaking (disassembling) of sets

with a finite number of elements into disjoint equipotent subsets (in cases when this is possible) (Milinković, 2022).

Present time textbooks' main characteristic is utilisation of multiple visual representations. A real context image with partitive division problems is a starting step in studying division. Notable, division is initially conceptualized procedurally, as a "repeated subtraction". For example, in a Serbian textbook (Milinković, 2019), in the first lesson on division, as seen in Figure 12, learners find out that this operation may be represented in widely different ways, from enactive in a realistic problem situation of equal sharing context of splitting 6 cakes among 3 children to iconic set representation (Vergnaud's multiplicative structure scheme), showing a connection between multiplication and division, to the number line further procedurally approaching division as a repeated addition (Milinković, 2019).

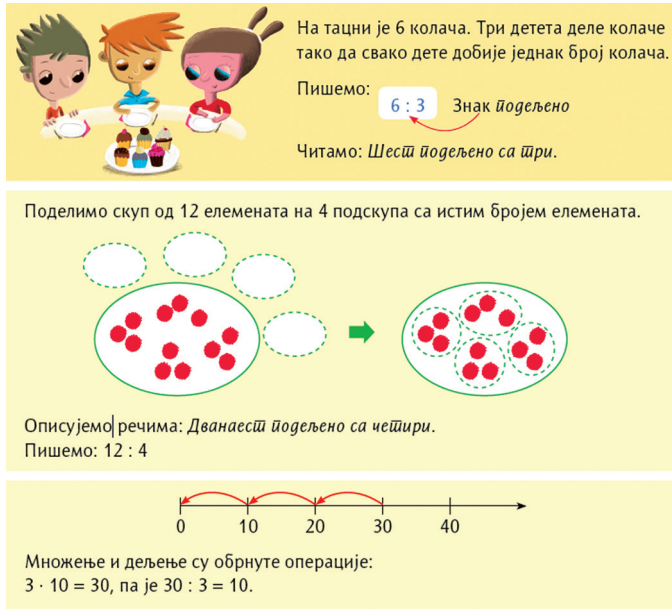


Figure 14. Multiple representations.

Transcript of Figure 14:
Frame 1: There are 6 cookies on the counter. Three children share the cookies so that each child gets an equal number of cookies. We write: 6 : 3*, "division sign",*
We read: Six divided by 3.
Frame 2: Divide a set of 12 elements into 4 subsets with the same number of elements. We describe in words: 12 divided by 4. We write: 12 : 4
Frame 3: Multiplication and division are inverse operations. $3 \times 10 = 30$ *so* $30 : 3 = 10$.

5. Discussion

We examined a selection of textbooks in a span of about 70 years used in the territory of the Central Balkans. In the 1960s, the arrays dominated in the conceptualization of division. It was meant to support development of understanding of

"measurement division" structures and provided a setting for creating a connection to multiplication. In the next decade, during the 1970s (New Math) a set representation was dominant with conceptual and procedural focus on the concept of division as a repeated subtraction. The following period, 1980s–1990s, brought into focus new visual representations: a number line and a bar. Gradually, focus was shifted from iconic representations to realistic contexts. The period from the 2000s to the present is characterized by multiple representations, the integration of real-life contexts, and conceptual and procedural focus of building connections between division and multiplication as opposite arithmetic operations.

The observed evolution of the didactical approach shows a shift from procedural to relational understanding. Theoretical frameworks help explain how different visual mediators privilege certain cognitive schemes. As Shulman (1986, p. 9) states that "there are no single most powerful forms of representation of ideas. . . ". One pedagogical implication of this analysis is that a variety of visual mediators better supports conceptual generalization.

6. Conclusions with implications

This study examined the visual introduction of division in primary school mathematics textbooks in Serbia and Croatia across more than seventy years. Focusing on first lessons and visual mediators, and guided by Vergnaud's multiplicative structures, Greer's classification of multiplicative situations, and Zweng's typology of division, the analysis revealed several key insights.

First, the choice of visualization has shifted historically, reflecting broader curricular trends. Arrays representation dominated in the early decades, the 1970s emphasized set-based and procedural approaches, the 1980s and the 1990s diversified into number lines, bars, and realistic contexts, while contemporary textbooks combine multiple representations.

Second, there has been a gradual move from procedural to relational understanding. Earlier emphasis on repeated subtraction ensured procedural fluency but limited conceptual depth. Modern approaches promote connections to multiplication and structural reasoning within the field of multiplicative thinking.

Some observed differences perhaps reflect local curricular reforms and publishing contexts but we did not attend to this in this study.

The implications are twofold. For textbook authors, the findings emphasize the importance of selecting and sequencing visual mediators that foster flexible reasoning across both partitive division and measurement division contexts. Integrating multiple complementary representations can better support conceptual generalization. For teachers, the analysis highlights the need to connect visual mediators explicitly in classroom practice, helping students move beyond mechanical computation toward deeper multiplicative reasoning.

Finally, this study points to directions for future research. Investigating how pupils actually interpret different visual mediators in practice, and how teachers

guide these interpretations, would provide valuable insights for instructional design. Further research could also explore the evolving role of teachers in mediating textbook representations across different historical and cultural contexts.

In conclusion, the evolution of division representations demonstrates that carefully chosen and connected visual mediators are central to building a deep, lasting understanding of division in primary mathematics education.

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Contact addresses:

Jasmina Milinković
Faculty of Education, University of Belgrade
Kraljice Natalije 43, 11 000 Beograd, Serbia
e-mail: milinkovic.jasmina@yahoo.com

Nives Baranović
Faculty of Humanities and Social Sciences, University of Split
Poljička cesta 35, 21 000 Split, Croatia
e-mail: nives@ffst.hr

Vizualizacija deljenja u matematičkim udžbenicima

Jasmina Milinković[1] i Nives Baranović[2]

[1]Fakulteta za obrazovanje učitelja i vaspitača, Sveučilište u Beogradu, Srbija
[2]Filozofski fakultet, Sveučilište u Splitu, Hrvatska

Sažetak.

Udžbenici matematike su osnovni resursi i za nastavnike i za učenike, oblikujući način na koji se matematički koncepti uvode i razumeju. Ova studija istražuje kako je koncept deljenja vizuelno predstavljen u udžbenicima za osnovne razrede istorijski. Konkretno, ispituje vrste vizuelnih reprezentacija koje su korišćene u uvodnim časovima deljenja u Srbiji i Hrvatskoj od 1950-ih do danas. Analiza je zasnovana na Vernovoj teoriji multiplikativnih struktura i Grirovoj klasifikaciji multiplikativnih situacija, pružajući okvir za procenu kako različite reprezentacije podržavaju konceptualni razvoj. Upoređivanjem istorijskih i savremenih pristupa, studija nastoji da identifikuje trendove, objasni razlike i proceni ulogu vizuelizacije u negovanju dubljeg razumevanja deljenja. Rezultati doprinose diskusiji o tome kako vizuelne reprezentacije mogu efikasno podržati učenje učenika i imati implikacije na dizajn udžbenika i didaktičke pristupe u matematičkom obrazovanju.

Ključne riječi: aritmetičke operacije, vrste deljenja, nastavni resursi, vizuelni medijatori

Designing Activities to Foster Primary Students' Geometrical Thinking Through Cognitive Apprehension

Branka Antunović[1] and Vida Manfreda Kolar[2*]

[1]Faculty of Educational Sciences, Juraj Dobrila University of Pula, Croatia
[2]Faculty of Education, University of Ljubljana, Slovenia

Abstract. This paper presents the practical application of Duval's theory of cognitive apprehension. The study analyzed classroom lessons designed to enhance geometrical reasoning of primary school students (aged 12–13). The lessons engaged students with polygons and circles through hands-on activities such as cutting, rearranging, and constructing figures using different tools. Observations revealed that students initially perceived geometric figures through perceptual, operative, and sequential apprehensions before transitioning to discursive apprehension, enabling them to infer properties and provide logical explanations. Teacher questioning and instructional guidance played a crucial role in shifting students from a perceptual view of figures to a more mathematical approach – coordinating all forms of geometrical figure apprehension. The findings emphasize the importance of learning environments where students actively engage with figures, allowing them to explore and manipulate shapes while strengthening their perceptual, operative, and sequential apprehension. Encouraging students to analyze and articulate their reasoning through dialogue further deepens their comprehension of geometric properties. Additionally, structured teacher interventions, particularly through questioning, facilitate the transition from exploratory interactions to more formal discursive reasoning.

By integrating these principles, the study contributes to bridging the empirical-deductive gap in geometry learning, reinforcing students' ability to reason geometrically and preparing them for argumentation in secondary education.

The results show that carefully sequenced, teacher-supported tasks grounded in Duval's framework can significantly enhance students' ability to engage in mathematical argumentation and reason geometrically.

Keywords: configurative process, geometrical reasoning, geometrical figure apprehension

*Corresponding author.

1. Introduction and theoretical background

Understanding geometry requires more than recognizing shapes – it demands a cognitive shift from visual perception to formal mathematical reasoning. Duval (2017) emphasizes that students often struggle to move beyond their initial perceptual impressions to identify the underlying mathematical structures of geometric representations, especially in tasks involving decomposition and reconfiguration (Duval, 2014).

Effective geometry instruction facilitates this cognitive progression by engaging students in both physical and mental manipulation of geometric figures. Duval's structured teaching sequences (2017) offer a developmental model for conceptual learning in geometry, emphasizing visualization and cognitive processing over computation. These sequences are particularly effective for introducing abstract geometric concepts by progressing from tactile exploration to formal reasoning.

In the first stage, students engage with physical objects – folding, rotating, or overlapping 2D and 3D figures – to explore geometric properties through action rather than measurement. The second stage shifts attention to analyzing visual representations, helping students identify and interpret basic geometric elements (e.g., points, lines) within larger configurations. This supports the development of abstract thinking. In the final, third stage, students articulate geometric relationships using formal language and terminology, linking earlier visual and physical observations to mathematical concepts and strengthening their reasoning skills.

Central to this pedagogical framework is Duval's (1995) theory of cognitive apprehension, which describes the mental operations students use to process and understand geometric figures.

He identifies four types of apprehension, each playing a distinct role in learning:

- Perceptual apprehension (PA): involves the visual recognition of geometric figures based on their appearance and spatial properties.
- Sequential apprehension (SA): refers to understanding geometric figures through the step-by-step processes used in their construction or analysis.
- Operative apprehension (OA): involves manipulation, transformation, or reconfiguration of figures, either physically or mentally, to explore properties, solve problems, or develop proofs.
- Discursive apprehension (DA): entails verbal or written reasoning using formal language, definitions, and theorems to justify or infer geometric relationships.

For a figure to function as a true geometric figure, PA must be combined with at least one additional type of apprehension, as perception alone does not suffice for mathematical reasoning (Duval, 1995, p. 143). Integrating these modes of apprehension enhances students' ability to think geometrically, promoting deeper understanding and flexible problem-solving (Bernabeu et al., 2021).

However, students frequently encounter difficulties in coordinating these modes – particularly in transitioning from perceptual to discursive reasoning –

resulting in fragmented understanding of geometric concepts. This gap becomes especially evident in secondary school education, where students are expected to articulate logical arguments and construct proofs. The contrast between intuitive, hands-on learning in primary school and the formal deductive expectations of secondary curricula often creates a discontinuity in learning (Fujita & Jones, 2014).

Bridging this gap requires early and intentional development of reasoning skills. Stylianides and Stylianides (2017) emphasize the importance of nurturing mathematical argumentation in primary school education, suggesting that foundational reasoning abilities developed at a young age can better prepare students for the demands of secondary-level proof and abstraction.

Duval's theory provides a valuable framework for supporting this development by promoting the integration of different apprehensions. Beyond visual recognition (PA), students must learn how to construct figures step-by-step (SA), manipulate them mentally or physically (OA), and express their ideas using precise mathematical language (DA). Coordinated use of these apprehensions fosters a deeper, more transferable understanding of geometry and supports students in constructing and justifying mathematical arguments.

Empirical studies consistently highlight the critical role of coordinating these cognitive processes. While PA allows for initial recognition, it must be supported by SA, OA, and DA to enable advanced reasoning and problem-solving. For example, SA strengthens understanding of geometric construction and supports deductive reasoning by helping students understand the logic of constructions step by step (Yang & Li, 2018), while OA supports creativity, such as through the insertion of auxiliary lines in proofs (Palatnik & Dreyfus, 2019; Gridos et al., 2022). OA has also been identified as a strong predictor of successful engagement in tasks with multiple solution paths (Palatnik & Dreyfus, 2019). DA, in particular, has been shown to serve as a central pivot in geometric reasoning, as it allows students to articulate, connect, and justify their thinking (Michael-Chrysanthou & Gagatsis, 2015; Deliyianni et al., 2009).

Proof construction requires a particularly close coordination between OA and DA. Torregrosa and Quesada introduced the concept of the configurative process to describe this interaction, defining it as "the set of mental operations undertaken by the student during the application of reason to geometry" (2008, p. 323), distinguishing it from communication-based processes. In this framework, OA enables students to structure and transform figures, while DA supports the logical justification of those transformations. Students who struggle with proof often fail to effectively coordinate these two types of apprehension. Since both are essential for constructing valid arguments, strengthening their interplay is key to fostering students' geometric reasoning (Llinares & Clemente, 2014, 2019).

Beyond proof, the integration of figure manipulation (PA/OA) and algebraic generalization also highlights the importance of coordinated apprehension in higher-level reasoning (Baranović & Antunović-Piton, 2021; Callejo et al., 2019). Recent models support a multimodal and nonlinear development of geometrical figure apprehension (GFA), consistent with Duval's (1998) view. High levels of DA are often linked to balanced development in PA and OA, suggesting that discursive

reasoning may play an integrative role across modalities (Michael-Chrysanthou et al., 2024).

Despite the importance of OA it remains underutilized in classrooms, often overshadowed by an overreliance on perceptual strategies (e.g., Michael-Chrysanthou & Gagatsis, 2014, 2015). Therefore, fostering students' ability to coordinate all four types of cognitive apprehension is essential for developing deep geometric understanding, creative problem-solving, and the capacity for logical argumentation skills that are critical for long-term success in mathematics.

2. Significance of the study

While Duval's theory of cognitive apprehension (1995, 2017) has been extensively studied across various educational levels – including its impact on students (e.g., Michael-Chrysanthou et al., 2024) and on pre-service and in-service teacher education (e.g., Llinares & Clemente, 2014) – there remains a notable gap in research focused specifically on its application in primary school classrooms. In particular, limited attention has been given to how teachers can integrate this theoretical framework into everyday instruction to foster young students' geometric reasoning.

This study addresses that gap by exploring the practical integration of Duval's framework into the design and implementation of geometry lessons in primary school education. It forms part of a broader collaborative action research project involving a researcher and practicing teachers, jointly investigating how classroom practices can be aligned with Duval's model to enhance students' conceptual understanding of geometry.

The focus here is on structured activities designed by teachers for 12–13-year-old students, which aim to strengthen geometrical thinking through the coordination of perceptual, sequential, operative, and discursive apprehension. By analyzing how these activities unfold in the classroom and how teachers guide students through cognitive transitions'from perceptual recognition to logical reasoning, this study offers practical insights into how theoretical models can inform instructional design.

More broadly, the study contributes to efforts to bridge the often-reported gap between intuitive, activity-based learning in primary school education and the more formal, argumentation-driven demands of secondary-level mathematics. By fostering the early development of reasoning and justification skills, it lays a foundation for long-term success in mathematical proof and deductive thinking.

3. Research question

This study addresses the following research question:

How can primary school geometry lessons be designed to develop students' reasoning and argumentation skills through the coordination of Duval's cognitive apprehensions?

4. Method

4.1. Participants

This study is part of a larger doctoral research project (Antunović, 2024) structured as collaborative action research (e.g. Betts et al., 2017; Bognar, 2011; Bonner, 2006). It involved four in-service mathematics teachers and their four seventh-grade classes (students aged 12–13) from urban schools in Istria County, Croatia. The experimental group consisted of two classes whose teachers were included in the *Development of Geometrical Thinking* (DGT) program, whereas the control group comprised two classes in which teachers taught in their usual way.

The focus of this paper is on the DGT program – a targeted instructional approach rooted in Duval's theoretical framework (see more in Antunović (2024)).

The program addressed core geometric content, particularly polygons and circles, and emphasized the integration of PA, SA, DA and OA in lesson design.

This paper specifically analyzes lessons implemented by two teachers in the experimental group, highlighting how they designed and facilitated activities aligned with Duval's theory to enhance students' conceptual understanding.

4.2. Data collection and analysis

The broader project employed a mixed-methods approach (Ivankova & Wingo, 2018), combining quantitative and qualitative data to evaluate the impact of the DGT program. While pre- and post-tests assessed progress in geometric thinking (quantitative), this paper focuses on the qualitative dimension, examining how cognitive apprehensions emerged during classroom instruction.

Data sources included: classroom observations and video recordings, researcher field notes and diaries, student notebooks and photographs of student work, teacher interviews and post-lesson reflections. To ensure reliability, data were triangulated across these sources. Thematic coding was applied, guided by Duval's four types of cognitive apprehension.

The thematic analysis focused on how these types of apprehension unfolded during instruction and how teacher interventions supported their coordination.

4.3. Classroom activities

In this section, we detail the implementation of teaching activities by experimental group teachers, as observed and analyzed through the lens of Duval's framework of GFA. These lessons were designed not only to teach core content about polygons and circles but also to develop students' reasoning, visualization, and argumentation skills. Five examples of lessons have been selected from the DGT program to demonstrate how teaching can foster geometrical thinking through the coordination of all four types of GFA.

Example 1: Thales' theorem

In lessons on Thales' theorem, students were guided to construct figures based solely on verbal descriptions, progressing through careful drawing and analysis. The teacher scaffolded the reasoning process using open-ended questions to shift student thinking from perceptual recognition of the figure (PA) to discursive (DA) and sequential apprehensions (SA). By adding auxiliary lines and interpreting symbolic notations, students engaged in deconstructing and reconstructing figures (OA) and justified properties like angle measures (DA), eventually proving that the inscribed angle above a diameter is a right angle. Below an excerpt from an extensive dialogue between a teacher and students is presented, illustrating how teachers facilitated the transition from figurative construction (SA) to formal thinking (DA), emphasizing the configuration process that is central to geometric reasoning.

ET: Ok, label it on the figure as α. Now, can you tell me what the measure of $\sphericalangle BSC$ is? Remember, the external angle of a triangle equals the sum of the two opposite interior angles (Figure 1a).

ES: Yes, the measure of $\sphericalangle BSC$ is 2α.

ET: Good. Now, in the same way, let's look at triangle SBC. Who can explain?

ES: Triangle SBC is isosceles because sides SB and SC are congruent – they are both radii of the circle. The angles at vertices B and C are therefore equal; let's call each of them β. The exterior angle at vertex S is then 2β.

ET: Ok, please write everything you just said on the board. *(ES writes on the board, see Figure 1b.)*

ET: Now that you've labeled the figure, what do you notice about the angle at vertex C?

ES: I see that the angle at C is the sum of α and β, and the angle at S is a straight angle.

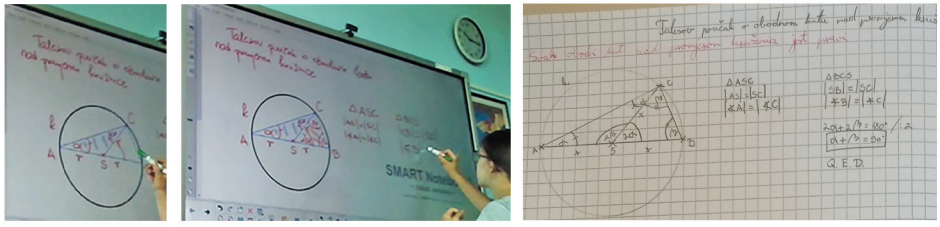
ET: Please write that down. *(ES writes on the board:* $2\alpha + 2\beta = 180°$. *Then says: 'That means* $\alpha + \beta = 90°$. *From the figure, we see that the angle at vertex C is* $\alpha + \beta$*, so that angle is a right angle.' (Figure 1c))*

ET: Excellent conclusion.

ET: So, the inscribed angle at vertex C, which lies on diameter AB, corresponds to the central angle at vertex S. Since an inscribed angle is half the measure of the corresponding central angle, it equals $90°$. This is exactly Thales' theorem, which is a special case of the inscribed angle theorem.

ET: Could we have explained it in another way? What do you think? Look at everything we labeled in the figure.

ES: We could also add up all the angles'



a) Teacher drawing b) Students' drawing and symbolical writing c) Students note

Figure 1. Activity of Thales' theorem of an inscribed angle.

It should be emphasized that this example involves mathematical content that is part of the secondary school curriculum (MZOM, 2019) which significantly exceeds the expected knowledge of seventh graders. Nevertheless, with appropriate guidance, teachers were able to address their students' discursive and operational apprehension (Figure 1c).

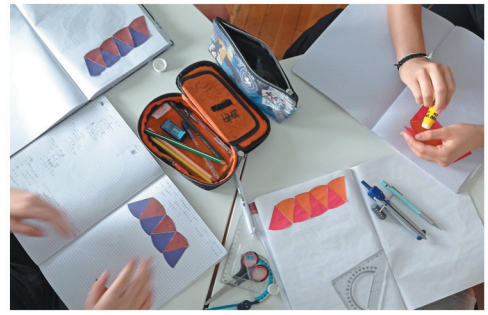
Example 2: Area of a disc through reconfiguration

A particularly illustrative lesson involved students constructing regular polygons inscribed in a paper disc. Students explored regular polygons and the circle by constructing, dividing, and rearranging paper models, following the next steps:

1. Constructing inscribed regular polygons. Each student brought a paper disc with a radius of 5 cm. Working in groups of four, they chose a type of regular polygon. The teacher asked guiding questions: *Which polygon do you like? Why? What do you know about it? How can you construct it?* Each group then inscribed their chosen polygon (e.g., triangle, quadrilateral, pentagon, hexagon, octagon, dodecagon) in the circle using radii and central angles (Figure 2a).
2. Dividing the disc into congruent sectors. Following the teacher's instructions, students cut their discs into equal 'pie' sectors. They exchanged some sectors with group members and rearranged them so that sectors alternated in color and matched along the radii. Finally, they pasted the new figure into their notebooks (Figure 2b).
3. From polygons to the area of a circle. Each group's representative presented their construction at the board, arranged by polygon type (starting with the triangle). The class then observed and discussed the figures (Figure 2c).



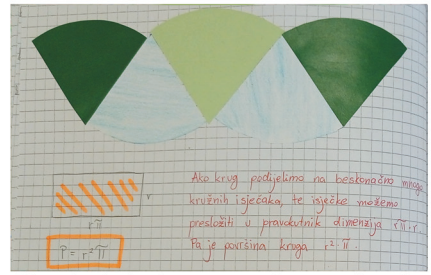
a) Step 1



b) Step 2



c) Step 3



If we divide the disc into an infinite number of circular sectors, we can rearrange these sectors into a rectangle with dimensions rπ and r. So, the area of the circle is $r^2\pi$.

d) Students note

Figure 2. Activities in the lesson: Area of disc through reconfiguration.

Through repeated manipulation and discussion, students developed a conceptual understanding of the area of a disc as a limiting case of rearranged polygon sectors (Figure 2d). These activities engaged students physically (OA), sequentially through construction (SA), and discursively as they articulated relationships between radius, circumference, and area (DA). The teacher's role was crucial in guiding inductive reasoning and connecting visual patterns to symbolic generalization.

Example 3: Tiling the plane and argumentation

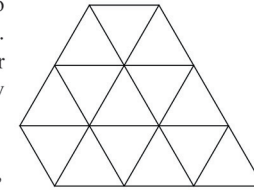
Lessons on tiling the plane served as a bridge between real-world phenomena and abstract mathematical conditions. The activity consisted of 4 steps:

1. Introduction to tiling the plane: The teacher introduced the concept with examples from daily life, nature, and art. Students discussed where tiling appears, related professions, and concluded that tiling (or tessellation) is a complete plane covering without gaps or overlaps, using geometric shapes edge-to-edge.
2. Conditions for tiling with regular polygons: Through a presentation, the teacher explained tilings with one or several regular polygons, introducing the idea of *hubs* (vertices). Students concluded that the interior angles at each vertex must sum to 360°, and polygons used together must have congruent sides.
3. Pair work: analyzing tilings: Using a worksheet (Figure 3a), students examined tilings with triangles, quadrilaterals, and hexagons, explained why these work, and provided counterexamples where tiling is impossible. They used both written and symbolic reasoning, identifying angles and vertices to support their arguments.
4. Group work: creating tilings: In groups, students used colored paper to construct at least two types of regular polygons, then arranged them into their own tilings. Finally, they explained and justified their choices, combining creative construction with clear geometric reasoning (Figure 3b).

These tasks combined PA, SA, and OA with DA as students provided written and verbal justifications about angle sums and side congruence, demonstrating the integration of knowledge across figures and the emergence of mathematical argumentation (Figure 3c).

Activity - worksheet - discovering the properties of the tiling the plane with the same figure

See the representation of tiling a plane with equilateral triangles. Try to explain why it is possible to tile a plane with congruent regular triangles. Prove by using the formulas about angles of polygons and give your argumentation in written form using as precise mathematical terms. Draw and label all you need on the pictures.

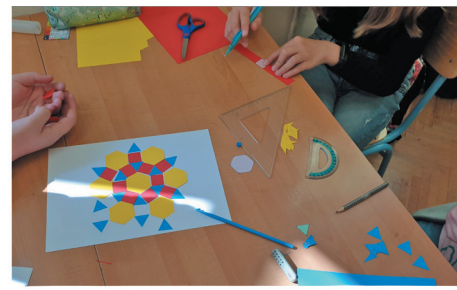


Choose any other regular polygon (except a regular triangle, quadrilateral, and hexagon), sketch it, and explain why it cannot be used to tile a plane, as in the above example.

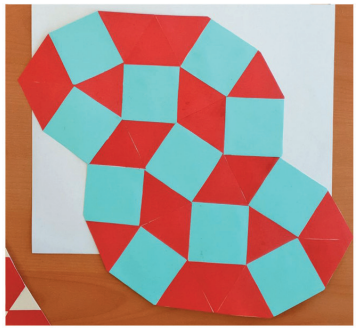
a) Step3.

Task for group work and students' construction of regular polygons and tiling the plane

Tile the plane with at least two regular polygons and explain why you can tile the plane this way.



b) Step 4.



Ravninu možemo popločati s bilo kojim pravilnim mnogokutima.

Važno je da su svakom mnogokutu stranice sukladne. Također uvijek mora iznositi 360° stupnjeva kako bi smo pravilno popločali ravninu.

We can tile the plane with regular polygons. The sides of each polygon must be congruent. The hub-vertex of tiling must be 360 degrees to properly tile the plane.

c) Students work.

Figure 3. Tiling the plane

Example 4: Perimeter and area of polygons and discs

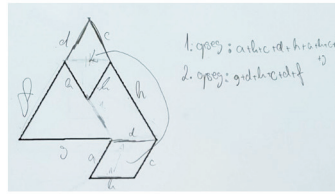
Through progressively complex tasks – ranging from reconfiguring a figure into an equilateral triangle to determining area and perimeter using square grids – students deepened their understanding of geometric properties. These lessons consisted of 3 activities:

1. Reconfiguring and comparing perimeters (Figure 4a).
2. Reconfiguring the figure using grid ' perimeter and area (Figure 4b).
3. Applying perimeter and area of disc – analyzing the given figure, drawing and determination of perimeter and area (Figure 4c).

These lessons emphasized symbolic notation (DA), grid-based constructions (SA), and reconfiguration strategies (OA). In particular, tasks without measurements required students to reason based on shape properties rather than calculations (Duval, 2014; Tůmová & Vondrová, 2017), promoting a conceptual understanding of area and perimeter and their distinction.

Reconfigure given figure
in the equilateral triangle.
Compare the perimeter of figures.

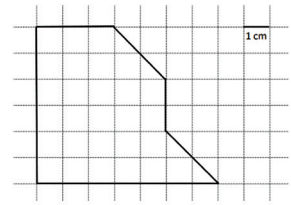






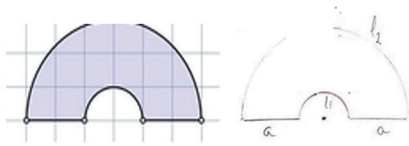
a) Activity 1.

Calculate the area of a polygon drawn in a square grid, in such a way that by "rearranging" its parts, you get a figure whose area you can determine.
Compare the perimeters of the initial and the "rearranged" figure.

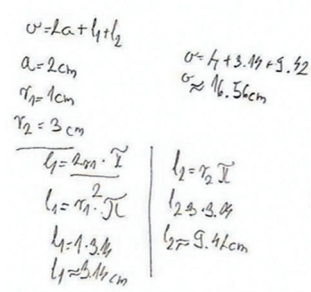


b) Activity 2.

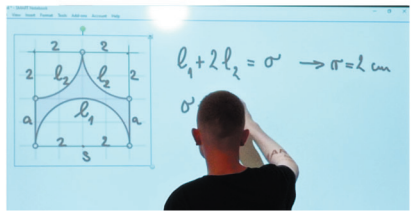
Determine the perimeter of the given figure in the square grid.
The side of unit square is 1 cm.



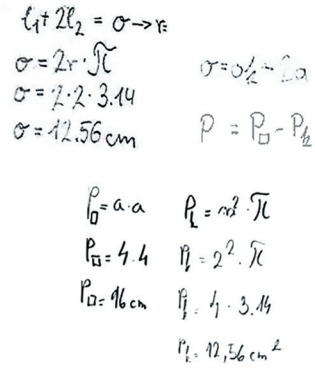
Task from the textbook and students drawing



Students note



Students labelling on the figure and solving on the smartboard



Students note

c) Activity 3.

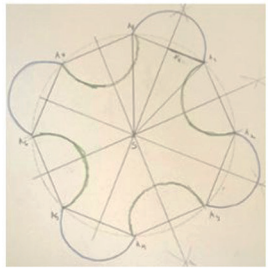
Figure 4. Perimeter and area of polygons and discs – qualitative tasks

Example 5: Integration tasks

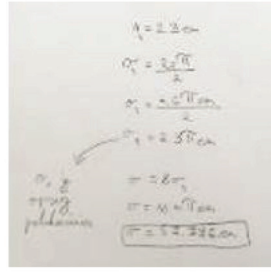
In summative tasks, students constructed complex figures combining regular polygons and circular arcs. They calculated perimeter and area, often without direct measurement, using deconstruction and symbolic reasoning. These tasks brought together all four types of apprehension in one activity: drawing (SA), manipulating subfigures (OA), recognizing patterns (PA), and articulating reasoning (DA). The culmination of learning demonstrated students' ability to coordinate apprehensions to solve non-trivial problems and explain their solutions clearly.

The teacher verbally gives all the construction instructions:

1. Draw a circle with a radius of 6 cm and inscribe a regular octagon inside it.
2. On each side of the octagon, construct a semicircle with the side as its diameter. Place the semicircles alternately: one outside the octagon, the next inside, and so on.
3. The resulting figure is the shape bounded by these semicircles (Figure 5a).
4. Calculate the perimeter of the constructed figure and explain the method used (Figure 5b).
5. Calculate the area of the figure and explain the method used (Figure 5c).

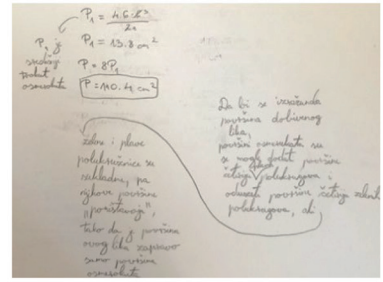


a) Resulting figure



O is perimeter of semicircles.

b) Perimeter



To calculate the area of the obtained figure, the area of the octagon can be added to the area of four congruent semicircles and the area of four semicircles subtracted, but the yellow and blue semicircles are congruent, so their areas are "cancelled", so that the area of the figure is equal to the area of the octagon.

c) Area

Figure 5. The students' solutions for an integration task.

5. Findings and discussion

This study explored how primary school geometry lessons can be designed to develop students' reasoning and argumentation skills through the coordination of Duval's cognitive apprehensions (PA, SA, OA, DA). Based on thematic analysis of classroom observations, student artifacts, and teacher reflections, several key findings emerged – highlighting the role of teacher interventions, the structure of learning activities, and the evolving nature of student reasoning.

1. A shift from perception to discursive reasoning – Across lessons, students consistently began with PA, visually recognizing shapes or patterns. However, without teacher mediation, their understanding remained superficial. Activities designed with SA and OA moved students beyond perception – encouraging construction, deconstruction, and mental reconfiguration of geometric figures. Ultimately, DA emerged through teacher-guided questioning and classroom dialogue, prompting students to explain, justify, and generalize.

2. The critical role of the teacher – The teacher's role proved pivotal in facilitating students' transition between types of apprehension. Teacher questions such as *'What do you notice?'*, *'Why is this angle* 90° *?'*, or *'Can you prove that?'* promoted metacognitive engagement and prompted students to connect visual, symbolic, and verbal reasoning. For example, during the lesson on Thales' theorem, the teacher gradually led students through a reasoning process with questions like:

   — *'What do you see?'*

   — *'What can you say now from this figure?'*

   — *'Can you explain with your own words what this expression means?'*

   These prompts supported DA by encouraging students to articulate relationships, use formal vocabulary, and engage in logical deduction.

   Similarly, in the lesson on the area of a disc, the teacher prompted OA by asking:

   — *'Now cut the disc into the marked sectors. What do you see when you rearrange them?'*

   — *'Imagine we constructed a 20- or 36-sided regular polygon. What would the shape look like?'*

   — *'And what is the length of this side?'*

   These questions enabled students to engage in inductive generalization and conceptual abstraction, building from concrete manipulation (OA) toward symbolic generalization (DA). It can also be seen that teachers scaffolded learning by guiding constructions (SA), encouraging reconfiguration (OA), and eliciting mathematical language (DA). Effective teacher prompts helped uncover sub-figures and latent properties within complex configurations, reinforcing the configurative process described by Duval.

3. Thematic analysis of classroom practices – Thematic analysis identified three major themes:

   — From concrete to abstract: Lessons moved systematically from physical manipulation of models (e.g., cutting discs) to abstract generalization (e.g., deriving area formulas), supporting Duval's instructional sequences.

   — Transformative interaction with figures: Tasks that required deconstructing and rearranging shapes activated OA and fostered spatial reasoning. This was evident in the disc area lesson, where sectors were rearranged into near rectangles, supporting algebraic generalization.

   — Language as a bridge: Verbal articulation played a central role in fostering DA. Teachers encouraged precise vocabulary and symbolic representation, pushing students toward formal reasoning.

4. Emergence of geometrical argumentation – Students showed growth in articulating and justifying geometric ideas. For example, when proving Thales' theorem or explaining tiling conditions, students used correct terminology and logical structures. Argumentation emerged not in isolation, but through coordinated tasks that built on PA, SA, and OA. Discursive reasoning was fostered through repeated prompts and peer discussions.

5. Sustained cognitive engagement – Tasks encouraged sustained engagement through challenge and relevance. Hands-on activities, real-world applications, and visual reasoning made abstract ideas accessible. Students were more invested when allowed to build, manipulate, and explain their models. Teacher encouragement of multiple solution paths also contributed to engagement.

   Following the DGT program, significant shifts were observed in both teachers' attitudes and teaching practices. Experimental teacher 1 (ET1) described how her focus had shifted from formulaic, calculation-based tasks to more interactive, exploration-driven methods. *'I now give tasks that allow students to explore geometric concepts through practical activities,'* she said, reflecting the program's emphasis on inquiry-based learning and critical thinking (Schoenfeld, 2019). Experimental teacher 2 (ET2) also reduced the reliance on technical tasks, opting instead for activities that encouraged students to manipulate shapes and explore geometric relationships. *'I have even done a lot of these tasks without concrete measures' and I think that's great,'* she noted, highlighting the success of removing rigid measurement-based tasks in favor of fostering deeper conceptual understanding. Both teachers observed significant improvements in student engagement following the implementation of hands-on and collaborative tasks. ET2 noted, *'There was no passiveness... I will do something because maybe I can do it too.'* This shift was particularly evident in group work, where students were encouraged to collaborate and share ideas. ET1 also observed, *'When they physically manipulate shapes, they better understand concepts like area and perimeter,'* which reinforced the importance of active, hands-on engagement in fostering a deeper conceptual understanding of geometry.

Additionally, both ETs acknowledged the value of theoretical frameworks in reshaping their teaching practices. ET1, in particular, found the research-based materials and activities useful, saying, *'The materials were inspiring and directly applicable in my classroom.'* However, ET2 emphasized the practical, interactive nature of the workshops, noting, *'I like more live, when you see or listen to someone else, and finally, practically when you work.'* Both teachers appreciated the theoretical underpinnings provided by the DGT program but felt the most immediate impact came from the practical application of these ideas in their classrooms (Bruce et al., 2011). ET2 also recognized the broader potential of action research, suggesting, *'Yes, definitely it would be useful' if there is a problem with how to convey it to them... why the problem is in my methods or is it really a problem that they are not mature or is something else a problem.'*

These findings confirm that the coordination of different apprehensions, scaffolded by teacher interventions, supports students not only in solving geometric problems but also in explaining, justifying, and generalizing their thinking, thereby answering the research question. Moreover, the observed progression from perceptual recognition to discursive articulation supports Duval's claim that geometric understanding depends on orchestrating multiple types of cognitive engagement (Duval, 2006). The empirical classroom data presented in this study offers concrete validation of this theoretical perspective.

Quantitative data supported these qualitative findings. In the experimental group, students showed statistically significant improvement in their coordination of geometrical figure apprehension (Wilcoxon, $p = .020$), whereas the control group did not ($p = .593$). This provides further evidence that the DGT approach, grounded in Duval's framework, not only engages students but leads to measurable gains in geometrical thinking (more about the quantitative research in Antunović (2024)).

6. Conclusion

This study provides concrete evidence of how Duval's theory of cognitive apprehension can be effectively integrated into primary mathematics instruction to develop students' reasoning and argumentation skills. By systematically engaging students in perceptual, sequential, operative, and discursive apprehensions (PA, SA, OA, DA), teachers created meaningful opportunities for deep mathematical learning.

The thematic analysis of the observed lessons indicates that teacher intervention was central to supporting students' progression from visual recognition to formal mathematical reasoning. Teachers functioned not merely as transmitters of content but as facilitators of cognitive development, employing structured questioning, strategic prompts, and dialogic teaching to scaffold students' construction and articulation of geometric ideas (Boaler & Brodie, 2004; Schoenfeld, 2019). These interventions were not isolated or incidental; rather, they were embedded within a carefully sequenced instructional design consistent with Duval's (2017)

framework. This design moved systematically from concrete manipulation to the identification of subfigures, and ultimately to the symbolic and verbal articulation of geometric properties and relationships.

The findings further suggest that the selection of appropriate learning activities, while necessary, is insufficient on its own. Equally critical is the teacher's capacity to recognize the pedagogical potential of each activity and to implement it in ways that foreground discussion, student reasoning, and guided exploration. Within the observed lessons, this approach enabled students to engage meaningfully with mathematical argumentation. Notably, students were able to follow and contribute to the construction of formal proofs by the seventh grade, addressing problems typically designated for more advanced levels of schooling or for students identified as gifted within the curriculum.

In sum, this study shows that effective implementation of Duval's framework is inseparable from intentional teacher facilitation. When supported by well-structured tasks, open-ended questioning, and opportunities for collaborative discussion, students are capable of developing rich geometric understandings. These insights underscore the importance of teacher professional development focused on both content knowledge and pedagogical strategies aligned with cognitive theories of learning.

In direct response to the research question, this study demonstrates that primary school geometry lessons can be designed to foster reasoning and argumentation by embedding Duval's framework into lesson planning. Through activities that sequentially activate PA, SA, OA, and DA – and through deliberate teacher facilitation – students gradually build towards formal geometric thinking. The framework proved both a diagnostic and pedagogical tool for improving students' cognitive engagement in geometry.

The findings highlight the importance of professional development grounded in theory and practice, reinforcing the transformative potential of the DGT program in reshaping both teaching practices and student learning outcomes. Our findings are in line with Obayashi and Fujita's (2025) identification of three essential elements for designing lessons that improve geometric reasoning in primary schools: (i) creating situations where students engage with diagrams in a generative way, thereby activating perceptual, operative, and sequential apprehensions; (ii) providing opportunities to determine logical explanations through classroom dialogue; and (iii) encouraging shifts toward discursive understanding through purposeful teacher questioning. The present study illustrates how these elements can be enacted in practice, particularly by linking operative to discursive apprehension as described in Torregrosa and Quesada's (2008) configurative process. Framing classroom design through this combined perspective strengthens the case for intentional teaching approaches that bridge the empirical'deductive gap between primary and secondary school geometry.

Future research should investigate how such approaches can be scaled across varied curricular contexts and how sustained teacher learning communities can deepen expertise in applying frameworks like Duval's to everyday classroom practice.

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Contact addresses:

Branka Antunović
Faculty of Educational Sciences, Juraj Dobrila University of Pula
Negrijeva 6, 52 100 Pula, Croatia
e-mail: bantunovic@unipu.hr

Vida Manfreda Kolar
Faculty of Education, University of Ljubljana
Kardeljeva ploščad 16, 1000 Ljubljana, Slovenia
e-mail: vida.manfreda-kolar@pef.uni-lj.si

Oblikovanje aktivnosti za razvoj geometrijskog mišljenja učenika osnovne škole kroz kognitivno razumijevanje

Branka Antunović[1] i Vida Manfreda Kolar[2]

[1]Fakultet za odgojne i obrazovne znanosti, Sveučilište Jurja Dobrile u Puli, Hrvatska
[2]Odjel za matematiku i računarstvo, Pedagoški fakultet, Sveučilište u Ljubljani, Slovenija

Sažetak. Ovaj rad predstavlja praktičnu primjenu Duvalove teorije kognitivnog razumijevanja. Studija je analizirala nastavu osmišljenu za poboljšanje geometrijskog rasuđivanja učenika osnovnih škola (u dobi od 12 do 13 godina). Nastavne lekcije uključile su učenike u rad s mnogokutima i kružnicama kroz praktične aktivnosti poput rezanja, preoblikovanja i konstruiranja figura pomoću različitih alata. Promatranja su pokazala da su učenici isprva percipirali geometrijske figure putem perceptivnog, operativnog i sekvencijalnog razumijevanja prije nego što su prešli na diskurzivno razumijevanje, što im je omogućilo donošenje zaključaka o svojstvima oblika i davanje logičnih objašnjenja. Pitanja učitelja i njegovo vođenje odigrali su ključnu ulogu u preusmjeravanju učenika s perceptivnog načina sagledavanja figura prema matematičkom pristupu, pri čemu su koordinirali sve oblike shvaćanja geometrijskih figura. Nalazi naglašavaju važnost okruženja za učenje u kojima se učenici aktivno bave figurama, omogućujući im istraživanje i manipulaciju oblicima dok jačaju svoje perceptivno, operativno i sekvencijalno razumijevanje. Poticanje učenika da analiziraju i jasno izraze svoje razmišljanje kroz dijalog dodatno produbljuje njihovo razumijevanje geometrijskih svojstava. Nadalje, strukturirane intervencije učitelja, osobito kroz propitivanje, olakšavaju prijelaz s istraživačkih interakcija na formalnije diskurzivno zaključivanje. Integracijom ovih načela istraživanje pridonosi premošćivanju između empirijskog i deduktivnog pristupa u učenju geometrije, jačajući sposobnost učenika za geometrijsko razmišljanje i pripremajući ih za argumentaciju u srednjoškolskom obrazovanju.

Ključne riječi: konfiguracijski proces, geometrijsko razmišljanje, razumijevanje geometrijskih figura

Applications of the Poly-Universe Game – an Experiment With Students

Edith Debrenti[1*] and Tünde-Klára Baranyai[2]

[1]Faculty of Economics and Social Sciences, Partium Christian University, Romania
[2]Babeş-Bolyai University Extensia Satu Mare, Romania

Abstract. The Poly-Universe is a game family, whose scheme is based on three basic shapes: triangle, square and circle, in red, green, yellow, and blue colours. The set is a very good motivational tool and an interesting didactic tool for lessons which can be integrated as a didactic game in almost any primary subject.

In our research, we will present an experiment that took place in the school year 2024/2025 with two groups of participants: high school students and teacher training university students from the Western Region of Romania (Oradea and Satu Mare). The experiment consisted of several sessions, focused on students designing activities for kindergarten and primary school children. Participants completed a standard Intrinsic Motivation Inventory questionnaire at the end of the session. The results showed that both high school students and university students came up with remarkably creative ideas, designing mathematical tasks that were embedded in realistic, real-life problem situations.

In our paper, we will present the results of the sessions, including their mathematical relevance, and process the responses to the questionnaires.

Keywords: mathematical education, Poly-Universe game family, non-digital games, game-based learning, STEAM education

1. Introduction

Geometric experience, spatial representation, spatial visualization, understanding the world around us, and developing the ability of spatial reasoning are fundamental aims in the teaching of mathematics (Freudenthal, 1972). According to Burger and Shaughnessy (1986), the van Hiele theory proposes that the development of

*Corresponding author.

geometric thinking and the acquisition of geometric knowledge occur through a series of hierarchical stages. In primary education, the emphasis is on the first two levels of the van Hiele model – recognizing figures (Level 1) and analysing figures (Level 2) – thereby establishing the foundational concepts of geometry (Debrenti, 2016). According to Szilágyiné (2013), the curriculum should promote the development of students' spatial visualisation and spatial reasoning skills, as well as their ability to recognise geometric figures in different contexts. Students should also be able to describe geometric figures and their properties accurately using appropriate geometric terminology. Herendiné (2003) argued that teachers should possess competencies in spatial thinking and spatial visualization, together with the ability to recognize geometric shapes in different contexts and describe them using appropriate geometric terminology.

According to Szendrei (2005, p. 319), "the involvement of hands and activities in reasoning is not a sign of lower order thinking, on the contrary, it means adapting to the opportunity to utilize abstraction in the topic in question". What can be solved by hand should be approached using concrete materials, as this method nurtures children's senses, memory, imagination, and spatial skills. Personal experience and practical activities are very valuable in education, especially in mathematics (Szilágyi et al., 2025).

Similarly, a core principle of Tamás Varga's complex mathematics teaching method was the use of real objects in the classroom to promote practical learning. His goal was to create mathematical models based on real-life problems and apply them in practice. An example of such a model is Poly-Universe game, a set of manipulative geometric shapes which encourages the exploration of geometric figures and their properties, supports the understanding of geometric concepts through visualization, and promotes active interaction, collaboration, discussion, and the development of students' geometric reasoning (Sousa et al., 2023). This study aimed to explore preservice teachers' use of Poly-Universe game when preparing educational activities.

2. Literature review

Recently, educators have begun using non-digital gamification (NDG) in schools as an alternative teaching method. NDG combines fun, exploration, and active participation through challenges, clear goals, immediate feedback, collaboration, and rewards (Ke et al., 2016). It employs physical tools like cards, dice, and board games, allowing students to solve problems, strategize, and discover independently (Park & Lee, 2017). This also helps teachers perform formative assessments by observing students during gameplay.

NDG has several advantages over digital games. It is affordable, requiring no technology setup, and is accessible to all students regardless of device availability. Its hands-on nature boosts engagement and understanding through multisensory experiences, which digital formats often lack. NDG encourages face-to-face interaction, fostering communication and teamwork, unlike the mostly individual screen

time of digital games. It also simulates real-world scenarios, linking theory with practice. By reducing screen time, NDG promotes a healthier learning environment. Overall, its flexibility, inclusiveness, tactile learning, and social focus make NDG a valuable educational tool (Setambah et al., 2024).

Game-based learning (GBL) should offer a variety of game types, as students' preferences, learning styles, and motivational factors vary (Szilágyi et al., 2026). Naderi and Moafian (2023) investigated the effects of digital and non-digital play-based instruction on children's vocabulary acquisition and retention. Their quantitative findings showed that the non-digital group consistently outperformed the digital group. Interviews and classroom observations supported these results, reinforcing their validity. This suggests that, despite technological advancements, the value of non-digital activities should not be overlooked. Further, Debrenti (2024a) found that although students may enjoy themselves and find task-solving engaging during both types of GBL, the use of digital and non-digital games can sometimes lead to different outcomes. Clements (1999) found that students who use manipulatives achieve better results in problem solving, as concrete experiences promote active engagement in the process of knowledge acquisition. Manipulatives also help students recognize relationships between concepts and enhance their skills in both problem solving and problem posing, even when dealing with tasks based on real-life situations (Rosli et al., 2015). Pálfalvi (2007) claimed that the online environment can never fully replace tangible, hands-on experiences.

As an example of an NDG, the Poly-Universe toolkit can be used in almost any classroom. The game supports experiential pedagogy and enables students to learn, collaborate, analyse, think, and problem-solve through play. Furthermore, activities based on the Poly-Universe are highly motivating for students (Sousa et al., 2023), which in turn enables motivated learning through play (Debrenti & Bella, 2025).

Acknowledging the teacher's role to implement experiential learning, it is important to promote the integration of GBL among educators (Vásquez et al., 2017).

Russo et al. (2021, 2024, 2025) found that over 75% of Australian primary teachers regularly use games – mainly NDG like board, dice, and card games – to support mathematics teaching. These games were seen as effective for developing fluency, understanding, problem-solving, and reasoning. Almost all teachers reported using math games weekly, with a strong preference for NDG. Only 1% chose digital games as their preferred classroom activity, and just 4% used any digital tools. Other studies confirmed that teachers are significantly more likely to use and prefer non-digital games over digital ones. Russo et al. (2025) attributed this preference to pedagogical advantages: better support for collaboration, hands-on learning, differentiation, and easier observation of student understanding.

Debrenti et al. (2025) surveyed 101 teachers from Finland, France, Hungary, and Romania and found that concrete manipulatives were widely used with children aged 3–10. Teachers saw these tools as helpful in easing the transition from kindergarten to primary school and valuable in both formal and non-formal settings. No significant differences were found based on gender or STEAM-related attitudes

and skills. Teachers emphasized the role of art in STEAM and the need for more practical tools and curricula.

In a qualitative research study, Palha and Jukić Matić (2023) discovered that the extent of pedagogical experience in utilizing games as an educational tool plays a pivotal role in determining teachers' receptiveness to game-based pedagogy. Vaz-Rebelo et al. (2025) stated that the role of teacher education programs in equipping future educators to use STEAM-based materials is often underemphasized. While many studies focus on student outcomes, fewer explore how teacher training supports the integration of such resources into formal education. Alshehhi et al. (2024) emphasized that effective implementation of GBL in university curricula requires deliberate efforts to develop subject-specific games that are both pedagogically sound and aligned with intended learning outcomes. The goal is to encourage preservice teachers to adopt GBL and gamification techniques in elementary classrooms, both during their practicum and in their future teaching careers. One way to achieve this is by introducing a university-level course that familiarizes students with the core GBL principles, gamification, and digital GBL, while also offering hands-on experience. Vankúš (2023) outlined a similar course structure and its activities, which could serve as a model for expanding the use of GBL and gamification in the preparation of preservice primary teachers.

3. Research methodology

3.1. Research questions

Following the previous research, our study focused on prospective teacher engagement with the Poly-Universe game. In their future work, preservice teachers will have to choose, create, and implement educational tools. Investigating their experience, engagement, output, and motivation during activity with Poly-Universe game was the main goal of this study (Debrenti, 2024b). For that reason, we posed the following research questions:

RQ1. What were characteristics of the games that students proposed? Did Poly-Universe game support mathematical activities?

RQ2. How did students perceive the use of NDG, particularly regarding their interest, engagement, and attitude?

RQ3. Were there significant differences in activities and affect between different groups?

3.2. The Poly-Universe game family

The didactic tool Poly-Universe is an educational game designed to introduce a new visual system for teaching mathematics (Santos et al., 2023). It offers multiple scientifically supported benefits and serves as a training system for both aesthetic and mathematical skills, grounded in the symmetry of scale change inherent in its geometric shapes. Composed of manipulable materials featuring combinations

of primary colours arranged in various ways, it allows for infinite combinations and aims to foster a novel methodology for mathematics education (Figure 1). By combining geometric shapes of different sizes and primary colours, it enables numerous mathematical explorations while fostering both aesthetic awareness and mathematical thinking. Simultaneously, the learning process engages multiple cognitive functions including vision, touch, detection, perception, attention, memory, discovery, and imagination, and mathematical abilities including abstraction, understanding of parts and wholes, recognition of relationships, identifying points of connection, creativity, and problem solving (Anđić et al., 2022).

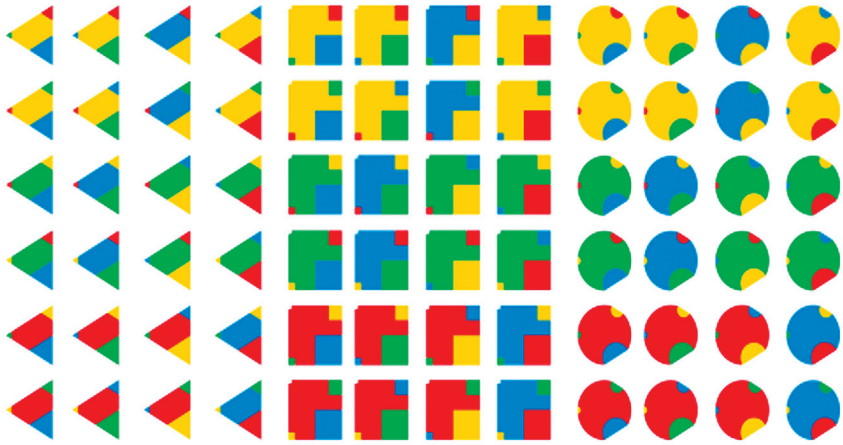

Figure 1. The elements of the Poly-Universe game family.

3.3. Participants

A total of 50 individuals were involved in the experiment. Our research took place in two different age groups: high school students specializing in teacher training and university students in teacher education from Romania's Western Region (Oradea and Satu Mare).

In the first group, 32 university students participated in the experiment, all girls. The students have already completed one semester of arithmetic, have enrolled in a mathematics didactics course during this study. They are familiar with the basic concepts used in the kindergarten and elementary school curriculum and they have prepared activities for both kindergarten and school groups. In the second group, 18 high school students (11th grade, pedagogy fields), participated in the experiment, 17 girls (94.44%) and one boy (5.56%).

Participants in this experiment worked in 4 groups: 2 groups worked on game proposals for children aged 3-6 (kindergarten) and 2 groups worked on game proposals for children aged 6-11 (primary school students).

3.4. Research design and procedure

From a methodological perspective, this study is exploratory (Swedberg, 2020), incorporating both qualitative and quantitative approaches based on data collected from observations and inquiry-based learning in classroom setting and questionnaires. The experiment was conducted between January and March during the 2024/2025 school year. It included multiple sessions in which participants were asked to design activities inspired by sample tasks, targeting kindergarten and primary school (Figure 2.A). Students were randomly assigned to groups and collaborated on game design proposals. The groups presented their ideas, which were documented, evaluated, and discussed, including possible variations (Figure 2.B).



A. Students designing activity *B*. Students presenting activity

Figure 2. Students using the Poly-Universe game.

At the end of the sessions, students completed a standard Intrinsic Motivation Inventory (IMI) questionnaire, developed by Ryan and Deci (2000). The questionnaire is a multidimensional measurement instrument designed to assess participants' subjective experiences related to a specific activity. It evaluates participants' interest/enjoyment, perceived competence, effort, perceived value/usefulness of the activity, pressure and tension experienced while performing the activity, and perceived choice during engagement in the activity.

We have used the following subscales of the original test: Interest/Enjoyment, Effort/Importance, Value/Usefulness. The questionnaire consisted of 16 statements related to the three mentioned dimensions. Students were asked to rate each statement on a five-point Likert scale (1 = not at all true; 5 = completely true) based on their experiences while participating in the activities involving the Poly-Universe tool. The quantitative data collected from student responses in the questionnaire were statistically analyzed using Excel.

4. Results

4.1. Game proposals

Although the kit is organized by shape, color, and size – suggesting a mathematical focus – the participants developed surprisingly creative and diverse activities. This section presents the analyses of the games that students proposed (Table 1), and provides several examples. Students' games were examined regarding the game objective, players, set (particularly the Poly-Universe toolkit), and rules.

Table 1. Games suggested by university and high school students.

Nr.	Title	Level	Topic	Authors
1	Adventure pool	Elementary class	mathematics	university students
2	Touch your nose! Clap! Hug!	Elementary class	skill game	high school students
3	Board game with steps	Elementary class	mathematics	high school students
4	Football field	Kindergarten	skill game	high school students
5	In the forest	Kindergarten	natural science	high school students
6	Construction of hexagons	Kindergarten	mathematics	university students
7	Group the circles	Elementary class	mathematics	university students
8	Creating 2D shapes from triangles	Elementary class	mathematics	university students, high school students
9	Forming digits from circles	Elementary class	mathematics	university students, high school students
10	Grouping by primary colour	Elementary class	mathematics	high school students
11	Dominoes	Kindergarten	mathematics	high school students
12	Colour operations	Elementary class	mathematics	high school students

Students used many concepts related to pre-mathematical skills such as size comparison (small, medium, large; smallest, largest), geometric shapes (shape, figure, square, triangle, circle, hexagon), their properties (centre, side, area, size, adjacency, intersection) and orientation, and number concepts and operations (number, quantity, point, score, addition).

Proposed games for primary school

Example 1. Adventure pool

- Game objective: recognizing colors and shapes
- Game players: class IV (10 years old), 2 groups
- Game set: Poly-Universe triangle shapes
- Game rules: make 4 hexagons with large triangles of the same color facing inwards. Set up a beach, where the cold-water pool is where the blue triangles meet, a playground is where the green triangles meet, a thermal

pool is where the red triangles meet, and a children's pool is where the yellow triangles meet (Figure 3).

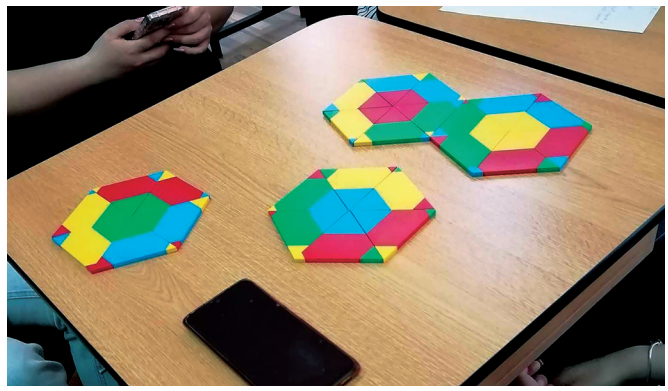

Figure 3. Adventure pool game.

Example 2. Touch your nose! Clap! Hug!

- Game objective: following instructions based on pictograms
- Game players: preparatory class (6-7 years old), 2 groups
- Game set: Poly-Universe square shapes (or triangle or circle shapes), pictograms with tasks
- Game rules: A track of square shapes is laid out, with squares of different colours and sizes chosen. On their backs are displayed different tasks with pictograms. They complete the selected task and then form a new track. For example, task for the small square is "Touch your nose!"; for the middle square: "Clap!", and for the big square "Hug!". After the end of the game, the group that obtained the highest score in the identification of figures wins, that is, for each correct answer the group wins 2 points, ending the game when one group reaches 10 points.

Example 3. Board game with steps

- Game objectives: recognizing colors, practicing addition
- Game players: elementary classes, 3-4 players
- Game set: Poly-Universe square (or triangle or circle) shapes, a figure for each player, modified dice
- Game rules: The elements of the set form a continuous playing field. We make a die containing the 4 colors (red, yellow, green and blue) on the fifth side there is a "miss" sign and on the sixth side there is a "roll again" sign. Players take turns rolling the dice, moving their figure to the first square on the color of the dice they roll. If the color is the base color of the square, they score 5 points, if it is the square with the largest area, they score 4 points, if it is the square of medium size, they score 2 points, and if it is the square with the smallest area, they score 1 point. They play until

they reach the end of the playing field. They add up the points they have collected and the one with the most points wins (Figure 4).

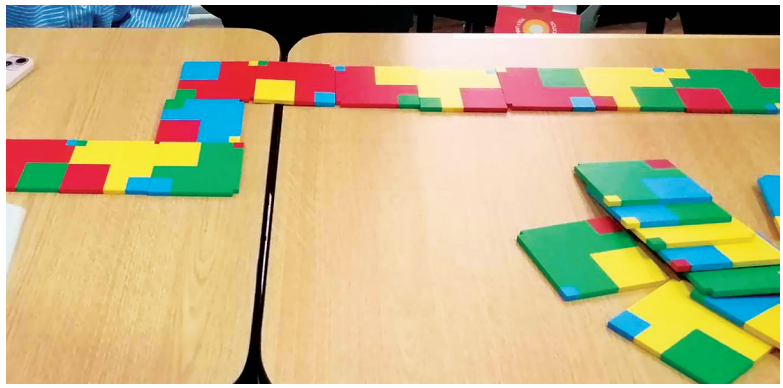

Figure 4. Board game with steps.

Proposed games for kindergarten

Example 4. Football field

- Game objective: matching numbers to quantities, developing fine motor skills
- Game players: kindergarten, 3-4 players
- Game set: Poly-Universe square shapes, dice, sponge ball
- Game rules: The children's task is to build a football field with the building materials. The children are given a dice and a small sponge ball. Their task is to throw the dice and roll the ball into the goal as many times as they throw the dice (Figure 5).

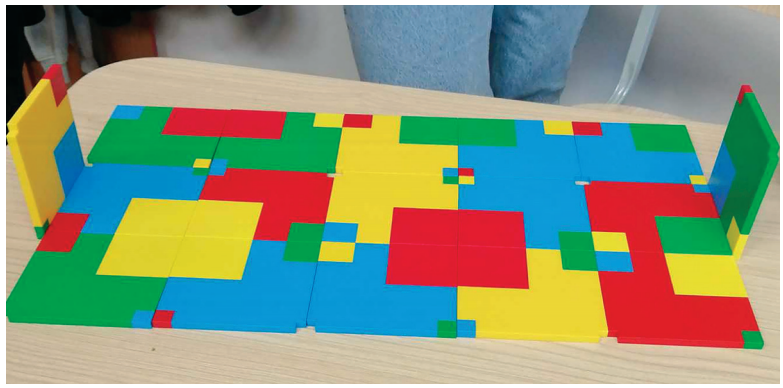

Figure 5. Football field.

Example 5. In the forest

- Game objective: recognizing and naming animals
- Game players: kindergarten, 3-4 players
- Game set: Poly-Universe square shapes
- Game rules: The children use the items in the set to create a forest, in which they place animals. The children walk through the forest and have to name the animals they encounter.

Example 6. Construction of hexagons

- Game objective: recognizing colours
- Game players: kindergarten, 3-4 players
- Game set: Poly-Universe triangle (or circle or square) shapes, modified dice
- Game rules: Using dice, each group is given a color and must put together a hexagon shape so that the center is the same color as the one they have been given.

Table 2. The mathematical concepts used in each game.

Example	Mathematical concepts used	Number of the concepts used	Authors
1. Adventure pool	triangle, hexagon, same, inward (spatial direction), group of four (4 hexagons)	5	university students
2. Touch your nose! Clap! Hug!	square, triangle, circle, size, small, middle, big, points, number (2 points, 10 points)	9	high school students
3. Board game with steps	square, triangle, circle, first, area, largest, middle, smallest, points, addition, end/start order	11	high school students
4. Football field	square, numbers, quantities, counting ("as many as")	4	high school students
5. In the forest	square	1	high school students
6. Construction of hexagons	triangle, square, circle, hexagon, center	6	university students

4.2. IMI questionnaire

During the intervention with the students, it was observed that the toolkit encourages group work, roles are formed almost automatically within the group, and the students encourage each other to move forward while working. The creation of something new and the freedom from making mistakes fostered their creativity. The analysis of students' responses to IMI questionnaire follows.

Table 3 presents the mean scores of the IMI questionnaire items related to participants' attitudes toward the activity for the whole sample, as well as separately for university and high school students. Overall, participants evaluated the activity

positively, with mean scores ranging from 4.06 to 4.12 for the positive statements, while the negatively worded item received a low mean score ($M = 1.64$). University students reported significantly more positive attitudes than high school students on all positive items ($p < 0.05$), whereas the difference between the two groups on the negatively worded item was not statistically significant ($p = 0.072$).

Table 3. Results of the IMI questionnaire: Attitude towards the activity.

Statements	Average	Average of university students	Average of high school students	Significance (2T tailed)
I really enjoyed this activity.	4.12	4.34	3.72	0.023
This activity was great fun.	4.12	4.44	3.56	0.001
This activity did not capture my attention at all.	1.64	1.47	1.94	0.072
I would say this activity was very interesting.	4.10	4.31	3.72	0.043
I think this activity was really enjoyable.	4.06	4.34	3.56	0.011

Table 4 presents the mean scores of the IMI questionnaire items related to participants' experiences during the activity for the whole sample, as well as separately for university and high school students. Overall, participants reported positive perceptions of their performance and engagement, with mean scores ranging from 3.30 to 3.92 for the positively worded items, while the negatively worded statements regarding lack of effort received a low mean score ($M = 1.92$). Although university students obtained slightly higher mean scores than high school students on most items, none of the differences between the two groups reached statistical significance (all $p > 0.05$).

Table 4. Results of the IMI questionnaire: Experiences during the activity.

Statements	Average	Average of university students	Average of high school students	Significance (2T tailed)
I think I did pretty well in this activity/course.	3.92	3.97	3.83	0.665
I think I performed well in this course compared to the others.	3.40	3.53	3.17	0.305
I am satisfied with my performance in the activity.	3.90	3.91	3.89	0.950
I put a lot of effort into this activity.	3.30	3.44	3.06	0.205
It was important for me to do well in the activity.	3.84	3.91	3.72	0.618
I did not put much effort into this course/activity.	1.92	1.84	2.06	0.506

Table 5 presents the mean scores of the IMI questionnaire items related to participants' perceptions after completing the experiment, both for the whole sample and separately for university and high school students. Overall, participants expressed positive attitudes toward the value and applicability of the Poly-Universe game, with mean scores ranging from 3.70 to 4.14 across the items. University students reported significantly higher ratings than high school students regarding the perceived value of the course, the ease of starting to use the Poly-Universe game, and their confidence in designing interdisciplinary activities with it (all $p < 0.05$). In contrast, no statistically significant differences were found between the two groups in their perceptions of using the game for teaching mathematics or its applicability across different disciplines (both $p > 0.05$).

Table 5. Results of the IMI questionnaire: Feelings after the experiment.

Statements	Average	Average of university students	Average of high school students	Significance (2T tailed)
I believe this course was valuable for me.	4.10	4.41	3.56	0.009
It was easy to start using the Poly-Universe game.	4.14	4.38	3.72	0.024
It will be easy for me to use the Poly-Universe game when teaching mathematics.	3.70	3.63	3.83	0.543
The Poly-Universe can be applied in various disciplines.	3.88	3.97	3.72	0.452
I know how to design and implement interdisciplinary activities using the Poly-Universe.	4.06	4.34	3.56	0.018

5. Discussion

Teachers can draw on a wide range of resources, from manipulatives to technological tools, to make lessons more engaging and increase students' interest in the content, and thereby support knowledge construction. The design and implementation of educational games that meet student needs and expectations to improve learning outcomes is an additional requirement for teacher education.

To answer the first research question, we looked at the activities suggested by university students and high school students according to their topics. Of the 12 games developed, 9 are math-related, 2 are skill-based, and 1 is related to natural science. University students, however, suggested only math-themed games, while high school students' game suggestions covered other topics as well. Participants of the experiment mostly suggested mathematics-related tasks and ideas for using the kit.

Not all games developed by the participants are presented in detail in this paper. However, based on the six game descriptions selected at random and included here,

the university students' two games contain a total of 11 mathematical concepts, corresponding to an average of 5.5 concepts per game. In comparison, the four games designed by the high school students contain a total of 25 mathematical concepts, corresponding to an average of 6.25 concepts per game. Although both groups suggested a variety of activities, several of their ideas were very similar or even nearly identical. Contrary to the expectation that university students would achieve significantly better proposals than high school students, there was no notable difference between the ideas or performance of university and high school students.

Regarding how students perceive NDG activities, it was observed during the intervention that they indeed showed a strong interest in such tools. They particularly liked the toolkit and enjoyed the activities associated with it. The toolkit effectively promotes group work; students demonstrated high levels of creativity, cooperation, and initiative. We additionally observed that university students were more engaged in every activity in this experiment.

Though all participants found the activities with the Poly-Universe kit to be useful, fun, effective, and enjoyable, there is a significant difference between the responses of university students and high school students regarding their attitudes towards the activity and their perceptions after it. Regarding experiences during the activity, we did not find a significant difference.

University students enjoyed the activities more ($p=0.001$), found the activity valuable ($p=0.009$), were confident in using the Poly-Universe game ($p=0.024$), and designing and implementing activities using game ($p=0.018$). The university students evaluated the activities more positively, especially items related to fun, enjoyment, and interest during the activity.

6. Conclusion

Students were critical of games created by their peers, citing issues such as unclear terms and the need to improve game rules. Overall, students engaged creatively in game play and documented the rules of their own games. They worked collaboratively, developing "new" games with creativity and thoughtful rulemaking. The activity demanded critical thinking and communication skills, which are essential today. Students performed above expectations, by generating and presenting many useful activities and ideas to their peers. Most of the activities/games proposed were mathematical activities.

Questionnaire results showed little variation in responses, but university students reported greater enjoyment and perceived value in the activity compared to high school students.

Future research directions include trying the activity with multiple groups, using several game sets simultaneously, implementing student-generated ideas in nurseries and schools, applying the tool in longer-term educational contexts.

Based on the results, we can conclude that such hands-on, tool-based activities are extremely beneficial for students, as students at different levels enjoyed them, found them useful, applied the knowledge they had learned while developing the games, and were able to collaborate effectively through group work.

The Poly-Universe toolkit is an innovative and creative game built on geometric elements, which means that students with strong mathematical skills tend to recognize shape relationships more quickly than those who struggle with math.

Through the presented playset, we aimed to demonstrate that the Poly-Universe toolkit can be effectively used in nearly any classroom setting. Playing games is an experiential activity, and it is through experiential pedagogy that students' competencies are most effectively developed. As learners engage in play, they collaborate, analyse, think critically, and solve problems. In essence, the teacher's primary role is to transform learning into an engaging experience. Play is the most effective way to motivate learners, and motivated learners are more likely to learn deeply and successfully.

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Contact addresses:

Edith Debrenti, PhD
Faculty of Economics and Social Sciences, Partium Christian University
Strada Primăriei 27, 410209 Oradea, Romania
e-mail: debrenti.edit@partium.ro

Tünde-Klára Baranyai, PhD
Babeş-Bolyai University Extensia Satu Mare
Petőfi Sándor 47, 440026 Satu Mare, Romania
e-mail: baratun@yahoo.com

A Poliuniverzum készlet alkalmazásai – beszámoló egy kísérletről

Edith Debrenti[1] és Tünde-Klára Baranyai[2]

[1]Gazdaság és Társadalomtudományi Kar, Partiumi Keresztény Egyetem Nagyvárad, Románia
[2]Szatmárnémeti Tagozat, Babeş-Bolyai Tudományegyetem, Románia

Kivonat. A Poliuniverzum egy játékcsalád, melynek formai és egyszerű színvilága három alapformára épít: a háromszögre, a négyzetre és a körre, színei a piros, a zöld, a sárga és a kék, A készlet nagyon jó motivációs eszköz, tanórák érdekes didaktikai eszköze, "játszva" minden tanuló sikerélményhez juthat. Az oktatást tekintve szinte bármely elemi osztályos tantárgyba beépíthető, mint didaktikai játék. Kutatásunkban egy kísérletet mutatunk be, amely a 2024/2025-ös tanévben zajlott. Tanító- és óvóképzős tanulóknak és egyetemi hallgatóknak mutattuk be a Poliuniverzum készletet. A kutatás mintáját Románia Nyugati régiójában két különböző csoport alkotta: pedagógia szakos középiskolai osztályok, illetve tanító- és óvóképzős hallgatók Nagyváradról és Szatmárnémetiből. A kísérlet több alkalomból állt, a résztvevőket csoportokra osztottuk és arra kértük őket, hogy a bemutatott tevékenységekhez hasonlókat találjanak ki, melyeket az óvodai és az elemi osztályos korosztály számára alkalmazhatóak. A résztvevők egy standard IMI (Intrinsic Motivation Inventory) kérdőívet töltöttek ki a foglalkozás végén. A csoportok bemutatták ötleteiket, ezeket dokumentáltuk és kiértékeltük, megbeszélve a lehetséges variánsokat is. Mivel az eszközöket könnyen osztályozni lehet alak, szín és méret szerint, kezdetben inkább egyszerű matematikai játékokra és feladatokra számítottunk, nem pedig érdemi kontextusba ágyazott, valós életből merített feladatokra. Az eredmények azonban meglepetést okoztak számunkra, mivel mind a középiskolás, mind az egyetemista diákok rendkívül kreatív ötletekkel álltak elő, és olyan matematikai feladatokat alkottak, amelyek valós, mindennapi problémahelyzetekbe ágyazódtak. Cikkünkben bemutatjuk a foglalkozások eredményeit, kitérve a matematikai jelentőségükre is, valamint feldolgozzuk a kérdőívek válaszait.

Kulcsszavak: matematikai oktatás, Poliuniverzum játékcsalád nem-digitális játékok, játékalapú tanulás, STEAM

Mathematics and Inclusion – The Role of Teaching Assistants in Working With Children With Difficulties in Mathematics

Ljiljanka Kvesić*, Katarina Šimić and Marina Zubac

Faculty of Science and Education, Department of Classroom Teaching,
University of Mostar, Bosnia and Herzegovina

Abstract. Inclusive teaching and inclusive education represent an approach within the education system based on the idea of equal opportunities for all students, regardless of their difficulties. Inclusive education emphasizes the importance of integrating students with difficulties while respecting their developmental diversity. Teaching assistants are individuals whose role is to provide support and assistance to teachers in conducting and organizing classes for students with developmental disabilities. The aim of this study was to examine the practices of teaching assistants in mathematics instruction for students with learning difficulties, focusing on their approach, methodological flexibility, and the application of adapted strategies. The study was conducted with a sample of eight teaching assistants, all female, using a questionnaire that addressed the use of teaching methods, emotional support, digital tools and educational games, as well as the promotion of student independence and self-confidence.

The results indicate that most assistants actively employ a variety of methods and strategies, adapt instructional content to students' abilities, and strongly encourage student autonomy and self-confidence. However, the use of digital tools and educational games is inconsistent, and all participants expressed a desire for additional resources and methods. The findings highlight the active and supportive role of teaching assistants in inclusive mathematics education and emphasize the need for ongoing professional development, availability of teaching resources, and formal recognition of their role in schools. These results contribute to understanding the importance of teaching assistants in inclusive education and may inform the development of policies and support programs in inclusive instructional practices.

Keywords: inclusive education, mathematics, teaching assistants

*Corresponding author.

1. Introduction

In contemporary education, increasing attention is devoted to creating an inclusive environment in which every child has the right to quality education, regardless of individual differences. Inclusion and inclusive education are based on the principles of equal opportunities for all students while respecting diversity. The goal of inclusive education is for every child to develop their potential in accordance with their own developmental abilities. The pedagogical and didactic practice of inclusive education aims to enhance the participation of all students not only in the learning process but also in the process of socialization. The concept of inclusive education manifests itself in different forms depending on the society and culture in which it takes place. Thus, inclusive education involves adapting teaching content, methods, and approaches, as well as ensuring support that enables the active participation of children with disabilities in mainstream classrooms (UNESCO, 2020).

In the context of teaching mathematics, an inclusive approach has particular significance, as mathematics is often perceived as a demanding and abstract subject that poses challenges for many students with learning difficulties. Children with difficulties in learning mathematics, especially those with dyscalculia, require structured, visually supported, and individualized assistance to develop an understanding of basic mathematical concepts and skills. Research shows that the use of different teaching methods, such as learning through play, practical examples, and visual aids, can significantly contribute to the success of students with mathematical difficulties (Geary, 2011; Butterworth, 2019). In this context, teaching assistants play a key role by providing direct support to students with disabilities, as well as to teachers in planning and implementing adapted activities. Teaching assistants act as a bridge between students and the learning process, helping them understand tasks, encouraging independence, and creating a positive environment. Their role is not only technical but also pedagogical, as it requires understanding the specific needs of students, applying adapted strategies, and collaborating with teachers and professional associates (Giangreco, 2010; Florian & Black-Hawkins, 2011). The aim of this paper is to examine the role of teaching assistants in the process of teaching mathematics to children with disabilities, to analyse their support strategies, and to consider how inclusive approaches contribute to students' achievement and motivation. The paper also seeks to highlight the importance of professional collaboration between teachers and assistants, as well as the need for continuous professional development in the field of inclusive pedagogy and mathematics didactics.

2. Theoretical background

2.1. Inclusive teaching and mathematics instruction

Inclusive teaching involves creating a school environment that enables the participation of all children, including students with developmental disabilities, in all aspects

of the educational process (Ainscow et al., 2006). The goal of inclusive education is not merely physical inclusion but also active participation and progress of students in accordance with their abilities. Inclusive teaching is directed equally toward students with and without disabilities. Inclusive educational practice and instruction entail the use of different teaching methods, flexibility toward students with disabilities, and culturally responsive pedagogy that includes all learners (Magee & Willey, 2024). Teaching mathematics in an inclusive context requires the use of various didactic approaches that match individual learning styles. In practice, this means that teachers plan and deliver instruction that is flexible, adaptable, and sensitive to students' individual differences. Inclusive teaching does not involve only the physical placement of students with disabilities in mainstream classrooms but also their active engagement, sense of belonging, and academic progress (Florian & Black-Hawkins, 2011).

In the field of mathematics, inclusive education presents a particular challenge because mathematics is abstract and requires the development of logical thinking, symbolic understanding, and problem-solving skills. Büscher and Prediger (2024) state that teachers engaged in inclusive mathematics practices face two key challenges: enabling joint learning for all students while simultaneously addressing the individual challenges of learners. Students with learning difficulties, especially those with dyscalculia, often show challenges in understanding mathematical concepts, recognizing relationships between numbers, and performing basic arithmetic operations (Butterworth, 2019). They require additional support through individualized teaching approaches, visual materials, manipulatives, and concrete examples that connect abstract mathematical ideas with real-life experiences (Geary, 2011).

Effective inclusive mathematics instruction is based on the principles of differentiated teaching and Universal Design for Learning, which enable access to content in various ways according to each student's needs. Teachers can thus apply multiple forms of content presentation (verbal, visual, practical), different types of tasks, and varied opportunities for demonstrating knowledge. Active participation in group activities and learning through play further supports the development of understanding and self-confidence in mathematics (Bouillet, 2010). Teaching assistants play an important role in the implementation of inclusive mathematics instruction. They provide direct support to students with difficulties in understanding tasks, encourage concentration and motivation, and help create a positive emotional climate in the classroom. Collaboration between the teacher and the assistant is essential for planning adapted activities that enable students to achieve learning outcomes in a manner appropriate to their abilities (Giangreco, 2010). In addition, assistants contribute to the development of students' social inclusion by enabling children to participate in shared activities with their peers. Given the complexity of mathematical concepts and the need for individualized approaches, inclusive mathematics instruction requires ongoing professional development for teachers and assistants, the exchange of experiences, and the development of an inclusive culture within the school. In this way, an educational system is created in which differences among students are recognized as an asset rather than an obstacle to learning.

2.2. Organization of mathematics instruction for students with developmental disabilities

The organization of mathematics instruction for students with developmental disabilities requires a systematic, thoughtful, and flexible approach that takes into account individual differences, abilities, and the needs of each student. At the core of planning lies the principle of individualized and adapted teaching, which enables students to access knowledge in a way that is understandable and achievable for them (Bouillet, 2010).

The first step in organizing instruction is the assessment of the student's abilities and difficulties. The teacher, in collaboration with the teaching assistant and the school's professional team (pedagogue, rehabilitation specialist, psychologist), analyses the student's strengths, interests, and areas requiring support. Based on this assessment, an individualized education plan is developed that includes adaptations of content, methods, forms of work, and assessment procedures (Kiš-Glavaš, 2011).

Mathematics content should be clearly structured and gradually built upon, with an emphasis on concrete and experiential learning. Abstract concepts need to be presented through concrete situations, objects, and visual supports (e.g., cubes, number lines, pictures, graphic representations). Students with difficulties achieve better understanding when examples from everyday life are used, such as adding money, measuring time, or comparing quantities (Butterworth, 2019). The teacher should use active learning methods such as play, pair work, group problem-solving, and multisensory approaches that integrate sight, hearing, and touch. In such an environment, students can explore and learn through experience, which promotes lasting understanding of mathematical concepts (Geary, 2011).

2.3. The role of the teaching assistant in inclusive education

Teaching assistants play an important supportive and facilitative role in inclusive education. Their primary responsibility is to provide individualized support to students during classroom instruction, encourage active participation, and promote students' independence. Teaching assistants collaborate with teachers in preparing and adapting learning tasks, explaining mathematical concepts, and monitoring students' progress. Kiš-Glavaš (2011) emphasizes that the key to effective support lies in a collaborative partnership between the teacher and the teaching assistant, with the focus on empowering students rather than fostering dependence.

In the context of mathematics education, teaching assistants can support students with dyscalculia by demonstrating problem-solving procedures, using concrete instructional materials (e.g., manipulatives, number lines, and graphic organizers), and providing emotional support and encouragement. The role of the teaching assistant in working with children with difficulties in mathematics is multidimensional and demanding. Teaching assistants contribute not only to students' cognitive development but also to their socio-emotional development. Through

collaboration with teachers and other educational professionals, they facilitate the implementation of high-quality inclusive practices. High-quality training and continuous professional support for teaching assistants are essential prerequisites for the successful implementation of inclusive mathematics education and for the development of schools that respect diversity and provide equal educational opportunities for all students.

3. Methods

The aim of this study was to examine in greater detail the role of teaching assistants in providing support to students with difficulties in learning mathematics within an inclusive educational setting. Particular emphasis was placed on the ways in which teaching assistants adapt instructional content and teaching methods, foster students' independence and self-confidence, and apply various instructional strategies and teaching resources in their everyday practice.

Furthermore, the study sought to identify the challenges teaching assistants most frequently encounter when working with students who experience difficulties in learning mathematics, as well as the strategies and examples of good practice they consider most effective for improving inclusive mathematics education.

In accordance with the aim of the study, the following research questions were formulated:

1. How do teaching assistants perceive their role in supporting students with difficulties in learning mathematics?
2. Which teaching methods, instructional strategies, and teaching resources do teaching assistants most frequently use when working with students with difficulties in learning mathematics?
3. To what extent do teaching assistants promote students' independence, motivation, and self-confidence during mathematics learning?
4. To what extent do teaching assistants use digital tools, educational games, and concrete teaching materials when working with students with difficulties in learning mathematics?
5. What are the most common challenges teaching assistants encounter when working with students with difficulties in learning mathematics?

3.1. Research participants

The study involved eight teaching assistants. All participants were female, reflecting the predominantly female representation in the teaching assistant profession. Therefore, when interpreting the results, it should be noted that the perspectives and experiences represent exclusively those of women, and, given the small sample,

the findings cannot be generalized. Table 1 presents the basic demographic and professional characteristics of the research participants.

Table 1. Basic demographic data of the research participants.

No	Sex	Age	Years of Experience	Education	Region
1.	Female	29	3	Pedagogy	WHC
2.	Female	34	5	Primary Education	HNC
3.	Female	30	4	Preschool Education	WHC
4.	Female	33	6	Pedagogy	WHC
5.	Female	35	8	Pedagogy	
6.	Female	39	5	Secondary School Education and Training for Teaching Assistants	WHC
7.	Female	43	2	Pedagogy	WHC
8.	Female	41	7	Secondary School Education and Training for Teaching Assistants	HNC

Within the research, a descriptive analysis of the basic characteristics of the participants, including age and years of work experience, was conducted. The aim was to obtain a clearer overview of the range of values and central tendencies describing the sample.

The analysis of descriptive statistics shows that the participants had an average age of 35.5 years $(\mathrm{SD} = 5.07)$, with the youngest participant being 29 years old and the oldest 43 years old. The median age is 34.5 years, indicating a relatively balanced age distribution within the sample.

Regarding work experience, the participants had an average of 5 years of experience $(\mathrm{SD} = 2)$, with a range from 2 to 8 years. The median work experience is also 5 years, showing that most participants have a moderate level of experience in their field. These results suggest that the sample includes participants with varying levels of professional experience, which may contribute to the diversity of perspectives in the study. Table 2 presents the minimum and maximum values, median, arithmetic mean, and standard deviation for the analysed variables.

Table 2. Descriptive statistics for age and years of work experience of the participants.

Analysed variables	Min	Max	Median	M	SD
Age	29	43	34.5	35.5	5.07
Years of Experience	2	8	5	5.0	2.0

3.2. Instrument

The research instrument consisted of a questionnaire developed specifically for this study, based on the instrument by Kizilelma et al. (2023), which was analysed and adapted to the context of teaching assistants in Bosnia and Herzegovina. The questionnaire included twelve closed-ended statements rated on a five-point Likert scale (1 – strongly disagree, 5 – strongly agree) and two open-ended questions aimed at exploring the personal opinions and suggestions of teaching assistants regarding the improvement of their role in mathematics instruction. The questions focused on the following thematic areas: the role of the assistant in teaching mathematics, collaboration with teachers and students, challenges in working with students with learning difficulties, and professional development and job satisfaction of the assistant.

3.3. Research procedure

The study was conducted in April 2025 in two counties, the Herzegovina-Neretva County and the West Herzegovina County, in Bosnia and Herzegovina. The participants were informed about the purpose and objectives of the research. Anonymity and confidentiality were guaranteed. Participation was voluntary, and the questionnaire was completed in writing. Completing the questionnaire took approximately 15 minutes. The collected data were used solely for the purposes of this research.

3.4. Data processing and analysis

Quantitative data collected through the Likert scale were analysed using descriptive statistics. Qualitative responses to the open-ended questions were analysed using qualitative content analysis, with thematic categories and recurring patterns identified in the participants' answers.

4. Results and discussion

The results obtained from surveying eight teaching assistants indicate a high level of engagement and positive attitudes toward the inclusive approach in teaching mathematics to students with developmental difficulties. Table 3 presents the frequencies and percentages of teaching assistants' responses to a series of statements related to their activities, strategies, and approach to teaching mathematics to children with learning difficulties.

The aim was to examine how assistants apply different methods and adaptations in teaching, how actively they promote students' independence and self-confidence, and how they use digital tools and educational games in their work with children. The results shown in Table 3 provide insight into the level of assistants' involvement in the teaching process, their methodological flexibility, and their motivation to use additional didactic resources in inclusive mathematics instruction.

Table 3. Frequencies and percentages of teaching assistants' responses to statements about mathematics instruction.

Statement	I completely agree	I agree	Neutral	I disagree	I completely disagree
1. I use various methods to explain a mathematical concept that is difficult for the student to understand.	n=6 (75.0%)	n=2 (25.0%)	n=0 (0.0%)	n=0 (0.0%)	n=0 (0.0%)
2. I believe that my role is crucial for the student's success.	n=6 (75.0%)	n=2 (25.0%)	n=0 (0.0%)	n=0 (0.0%)	n=0 (0.0%)
3. I approach students who are afraid of math with patience and encourage them to overcome their fear.	n=3 (37.5%)	n=3 (37.5%)	n=2 (25.0%)	n=0 (0.0%)	n=0 (0.0%)
4. I use adapted strategies when a student has difficulties with basic arithmetic operations.	n=6 (75.0%)	n=2 (25.0%)	n=0 (0.0%)	n=0 (0.0%)	n=0 (0.0%)
5. I adapt math lessons according to the abilities and needs of students with difficulties.	n=7 (87.5%)	n=1 (12.5%)	n=0 (0.0%)	n=0 (0.0%)	n=0 (0.0%)
6. I encourage the student to achieve greater independence in solving math problems.	n=6 (75.0%)	n=2 (25.0%)	n=0 (0.0%)	n=0 (0.0%)	n=0 (0.0%)
7. I actively work on fostering the student's self-confidence in learning mathematics.	n=6 (75.0%)	n=2 (25.0%)	n=0 (0.0%)	n=0 (0.0%)	n=0 (0.0%)
8. When a student shows frustration, I respond in a way that encourages engagement.	n=4 (50.0%)	n=3 (37.5%)	n=1 (12.5%)	n=0 (0.0%)	n=0 (0.0%)
9. I use digital tools that help students master mathematical content.	n=4 (50.0%)	n=2 (25.0%)	n=2 (25.0%)	n=0 (0.0%)	n=0 (0.0%)
10. I use educational games in math lessons.	n=6 (75.0%)	n=0 (0.0%)	n=2 (25.0%)	n=0 (0.0%)	n=0 (0.0%)
11. I incorporate concrete materials (e.g., blocks, sticks, and number lines) in working with students.	n=6 (75.0%)	n=0 (0.0%)	n=2 (25.0%)	n=0 (0.0%)	n=0 (0.0%)
12. I would like to use additional tools and methods in teaching mathematics for children with difficulties that I currently do not have at my disposal.	n=8 (100.0%)	n=0 (0.0%)	n=0 (0.0%)	n=0 (0.0%)	n=0 (0.0%)

The data analysis clearly shows that teaching assistants play an active and proactive role in inclusive mathematics instruction. High percentages of cases in which assistants "strongly agree" with statements regarding methodological flexibility, adaptation of strategies, and fostering students' independence and self-confidence indicate their awareness of the importance of an individualized approach for students with learning difficulties. For example, 75% of assistants agree or strongly agree that they use different methods to explain mathematical concepts to students who find them challenging, suggesting a well-developed ability to adapt teaching styles to students' needs.

Additionally, almost all assistants (87.5%) strongly agree with the statement that they adapt instructional units according to students' abilities. The results confirm their awareness and active involvement in individualizing and differentiating instruction. Supporting students' independence (75%) and self-confidence (75%) further demonstrates that their role goes beyond mere assistance. They serve as both emotional and pedagogical support, not just as helpers in completing tasks.

However, the data also reveal certain challenges. For instance, regarding the statement that they respond to students' frustration by encouraging engagement and motivation, only 50% strongly agree, 37.5% agree, and 12.5% are neutral. This may indicate that some assistants do not have a consistent strategy for handling student frustration or may need additional support in developing emotional and motivational skills. Interesting results were also observed regarding the use of digital tools. Half of the teaching assistants strongly agree that they use computer programs and tools for mathematics, while 25% are neutral and 25% report not using such tools. These results suggest uneven implementation of digital support. Similarly, with educational games, 75% of assistants fully use them, while 25% responded neutrally. This may indicate that, although there is strong motivation for innovative methods, not all assistants are equally equipped or trained to implement them.

One particularly important finding is that 100% of respondents strongly wish to use additional tools and methods that they currently do not have access to. This clearly demonstrates their high professional motivation and awareness of the limitations of existing resources. This finding is significant because it implies the need for institutional support (from school administration, ministries of education, or other authorities) to provide materials, training, and structures that enable assistants to fully develop their potential.

4.1. Analysis of open-ended question 1: the greatest challenges in working with students with learning difficulties in mathematics

Analysis of responses to the open-ended question, "What is your greatest challenge in working with students with learning difficulties?" reveals several thematic categories that reflect the complexity and diversity of the teaching assistants' roles in inclusive education.

The most frequently mentioned challenges relate to students' specific difficulties in mastering basic mathematical operations, particularly multiplication

and division. These areas require a higher level of abstract thinking, which can pose significant problems for children with developmental difficulties, especially those with dyscalculia. Assistants emphasized the need for additional adaptations and didactic materials that would enable students to engage in more concrete and experiential learning.

Some participants highlighted difficulties in explaining instructional content and adapting the curriculum to individual student needs, indicating a challenge in finding an appropriate balance between the curriculum requirements and students' abilities. It is particularly noteworthy that one comment mentioned the crucial support of an educational rehabilitator, underscoring the importance of interdisciplinary collaboration for successful inclusive instruction. Several participants also pointed out a lack of support and professional training for working with children with learning difficulties in mathematics. This finding aligns with previous research showing that assistants often face a shortage of systematic training, professional guidance, and clearly defined roles in the teaching process (Harris & Sass, 2011; Florian & Black-Hawkins, 2011).

Interestingly, some participants noted students' lack of motivation and interest in mathematics as a key challenge. This insight highlights the emotional-cognitive dimension of learning mathematics. Many students with learning difficulties develop fear or resistance toward mathematical tasks, which complicates the teaching process itself. Overall, participants' responses indicate that the greatest challenges arise from a combination of students' cognitive difficulties, limited resources, and lack of professional support, with assistants often attempting to compensate for systemic gaps through their own effort and adaptation. The results emphasize the need for continuous professional development for assistants and improved organization of team support within the school.

4.2. Analysis of open-ended question 2: best practices and advice for supporting students in learning mathematics

Analysis of responses to the question, "What is your best practice or advice for supporting students in learning mathematics?" shows that teaching assistants recognize the importance of motivational and experiential approaches to learning mathematics. The most frequently mentioned strategies include learning through play, using colourful and shaped materials, exercising patience, and creating a positive and relaxed atmosphere in which students can approach mathematical tasks without fear.

Several participants emphasized the need to alleviate students' fear of mathematics and make the content engaging, which aligns with previous research highlighting the importance of emotional support and creating positive experiences in mathematics learning (Boaler, 2016). Some participants also stressed the importance of connecting mathematics with logical thinking, practice, and repetition, reflecting awareness of the need for a gradual and structured approach to learning. Some responses also indicated the importance of individualizing instruction, such as adapting content for students who struggle with calculations by using various

aids and visual materials. These responses reflect an understanding of inclusive principles and the significance of using concrete didactic resources for students with developmental difficulties.

However, it is notable that some recommendations (e.g., "regular practice" or "connecting with logic") are more general in nature and may be more suitable for typically developing students than for those with pronounced learning difficulties. This may suggest a limited understanding of the specific needs of students with mathematical difficulties, which is often associated with a lack of specialized training and professional support for teaching assistants.

Overall, the responses show that assistants understand the importance of emotional support, positive motivation, and concrete materials in their work. At the same time, they reveal the need for further professional training in specific strategies for teaching children with difficulties in mathematics. Systematic education and mentorship for assistants could contribute to greater effectiveness and more precise recognition of students' needs in inclusive mathematics instruction.

5. Conclusion

The results show high levels of agreement, which indicate a positive orientation of assistants toward an inclusive approach, which is an important indicator of their readiness and awareness of their role in empowering students. However, the strong agreement across almost all statements may mask the real challenges assistants face in practice, such as lack of professional support, limited resources, insufficient collaboration with teachers, or a lack of continuous training in mathematics and specific learning difficulties.

It is particularly noteworthy that lower results were recorded in the use of digital tools, which may reflect the actual situation in schools – lack of technical resources, lower digital competence, or absence of specialized programs for students with mathematical difficulties. This dimension opens opportunities for future research and the development of training aimed at strengthening assistants' digital literacy.

A limitation of this study is that it was conducted on a small and homogeneously structured sample of exclusively female teaching assistants from a limited geographical area, which restricts the generalizability of the results. The data are based on self-assessment, without objective measures or tracking of student outcomes. Future research should include a larger and more diverse sample, adopting a longitudinal approach that combines self-assessment with classroom observation and analysis of student results. Schools and educational authorities should provide additional materials, digital tools, and continuous training for assistants, as well as formalize their role in lesson planning and evaluation.

In conclusion, although there is a possibility that responses were partially shaped by expectations of the "ideal" assistant behavior, the results still provide

valuable insights into their professional attitudes and areas where additional support would be beneficial – particularly regarding the use of modern technologies, material resources, and innovative didactic approaches.

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Contact addresses:

Ljiljanka Kvesić
Faculty of Science and Education, University of Mostar
Matice hrvatske bb, 88000 Mostar, Bosnia and Herzegovina
e-mail: ljiljanka.kvesic@fpmoz.sum.ba

Katarina Šimić
Faculty of Science and Education, University of Mostar
Matice hrvatske bb, 88000 Mostar, Bosnia and Herzegovina
e-mail: katarina.simic@fpmoz.sum.ba

Marina Zubac
Faculty of Science and Education, University of Mostar
Matice hrvatske bb, 88000 Mostar, Bosnia and Herzegovina
e-mail: marina.zubac@fpmoz.sum.ba

Matematika i inkluzija – uloga asistenata u nastavi u radu s djecom s teškoćama u matematici

Ljiljanka Kvesić, Katarina Šimić i Marina Zubac

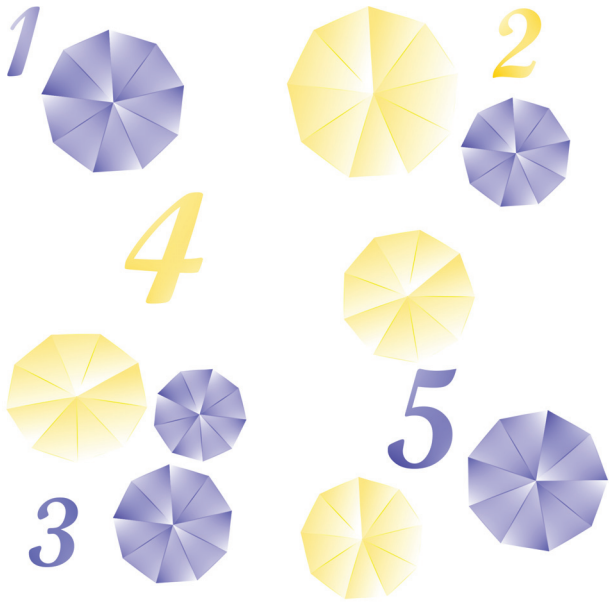
Fakultet prirodoslovno-matematičkih i odgojnih znanosti
Sveučilište u Mostaru, Bosna i Hercegovina

Sažetak. Inkluzivna nastava i inkluzivno obrazovanje predstavljaju pristup u odgojno-obrazovnom sustavu koji se temelji na načelu jednakih mogućnosti za sve učenike, bez obzira na njihove teškoće. Inkluzivno obrazovanje naglašava važnost uključivanja učenika s teškoćama uz poštivanje njihove razvojne različitosti. Asistenti u nastavi osobe su čija je uloga pružanje podrške i pomoći učiteljima u provedbi i organizaciji nastave za učenike s teškoćama u razvoju. Cilj ovog istraživanja bio je ispitati praksu rada asistenata u nastavi matematike s učenicima koji imaju teškoće u učenju, s posebnim naglaskom na njihov pristup radu, metodološku fleksibilnost i primjenu prilagođenih strategija. Istraživanje je provedeno na uzorku od osam asistentica u nastavi primjenom upitnika kojim su ispitani korištenje nastavnih metoda, pružanje emocionalne podrške, primjena digitalnih alata i edukativnih igara te poticanje samostalnosti i samopouzdanja učenika.

Rezultati pokazuju da većina asistenata aktivno primjenjuje različite metode i strategije rada, prilagođava nastavne sadržaje sposobnostima učenika te snažno potiče njihovu samostalnost i samopouzdanje. Međutim, primjena digitalnih alata i edukativnih igara nije ujednačena, a sve sudionice izrazile su potrebu za dodatnim nastavnim sredstvima i metodama rada. Dobiveni rezultati potvrđuju aktivnu i podržavajuću ulogu asistenata u inkluzivnoj nastavi matematike te naglašavaju potrebu za kontinuiranim stručnim usavršavanjem, osiguravanjem odgovarajućih nastavnih resursa i formalnim prepoznavanjem njihove uloge u školama. Rezultati istraživanja pridonose boljem razumijevanju važnosti asistenata u inkluzivnom obrazovanju te mogu poslužiti kao polazište za razvoj obrazovnih politika i programa podrške u području inkluzivne nastavne prakse.

Ključne riječi: inkluzivno obrazovanje, matematika, asistenti u nastavi

3.
Digital Learning Environments, Game-Based Learning, and Assessment in Mathematics Education



Students' Perceptions of the Game GeomWiz: Analysis of Digital Game-Based Learning in Secondary Mathematics

Ljerka Jukić Matić[1]* and Darija Marković[2]

[1]School of Applied Mathematics and Informatics, Josip Juraj Strossmayer University of Osijek, Croatia
[2]Faculty of Civil Engineering and Architecture Osijek, Josip Juraj Strossmayer University of Osijek, Croatia

Abstract. Digital game-based learning is increasingly recognized as an effective approach to teaching mathematics, demonstrating a positive impact on students' academic achievement, motivation, problem-solving skills, and self-confidence. While its use is widespread in primary education, integration into secondary education remains limited. One contributing factor is the lack of digital games specifically designed to address the more abstract and complex mathematical concepts introduced at the secondary level. To address this gap, digital games and corresponding teaching scenarios were developed within the Erasmus+ project GAMMA, with a focus on secondary school mathematics topics such as geometry. The purpose of this study is to investigate students' attitudes toward GeomWiz, as well as their overall experiences with the associated instructional activities. Data were collected through a questionnaire that combined closed-ended and open-ended items, administered to students in Finland and Croatia. Descriptive statistics were used to analyze the closed-ended responses, while the open-ended responses underwent qualitative analysis. The findings offer valuable insights into student engagement, perceptions of DGBL, and the potential of such tools to enrich secondary mathematics education.

Keywords: digital game-based learning, digital games, secondary school mathematics, students, GeomWiz

*Corresponding author.

1. Introduction

Digital game-based learning (DGBL) has attracted growing attention in mathematics education as a way to address one of the field's persistent challenges: students' lack of motivation and engagement. Mathematics is often perceived as abstract, difficult, or disconnected from everyday life, which contributes to declining interest and limited participation. By embedding mathematical concepts into interactive environments, DGBL has the potential to transform learning into a more enjoyable and meaningful experience, with studies reporting positive effects on motivation, persistence, and attitudes toward mathematics (Fadda et al., 2022; Hussein et al., 2022). Despite these encouraging findings, most research has focused on single educational contexts, with little attention to how students from different cultural and curricular backgrounds experience DGBL. Yet, perceptions of game-based learning may vary depending on how mathematics is taught, how students are accustomed to engaging with digital tools, and what value they attribute to games in school settings. Understanding these differences is key for identifying both the potential and the limitations of DGBL in diverse classrooms.

This study addresses this gap by examining how secondary school students from Croatia and Finland perceived a mathematics lesson that incorporated the educational game GeomWiz, developed within the Erasmus+ GAMMA project. Specifically, we ask: How do students from different countries evaluate a DGBL lesson compared to their regular mathematics lessons? By exploring both supportive and hindering aspects of students' experiences, the study aims to provide insights into the pedagogical opportunities and contextual challenges of implementing digital games in mathematics education.

2. Study context: GAMMA project and GeomWiz game

This study was conducted within the framework of the GAMMA (Game-Based Learning in Mathematics) project, a three-year Erasmus+ initiative under the Strategic Partnerships for School Education program. The project brought together researchers, teacher educators, and secondary school mathematics teachers from universities and schools in Greece, Croatia, the Netherlands, and Finland. Its main goal was to design and implement digital games specifically tailored for secondary school mathematics, as most existing educational games were developed primarily for younger learners.

Among several games developed, GeomWiz was selected for implementation in this study. GeomWiz (Figure 1) is a single-player educational mobile game designed to strengthen mathematical understanding through interactive practice. Players progress through levels of increasing difficulty, each associated with specific learning objectives. Levels consist of a series of rooms with questions rated from one to four stars, where more challenging questions yield higher points. To advance, students must answer at least one question correctly, with optional hints

and feedback provided after incorrect answers. At the end of each level, players receive performance statistics and a trophy, while their cumulative progress is tracked on a diploma platform.

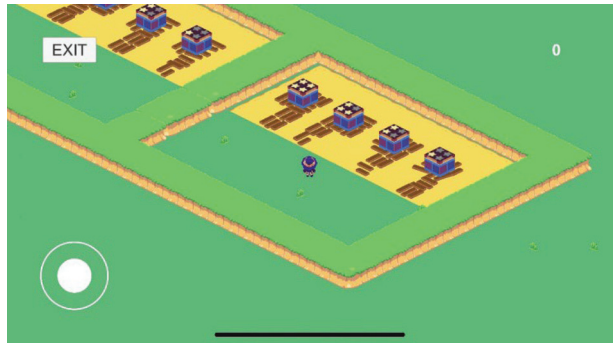


Figure 1. GeomWiz Game.

One of the GAMMA project's aims was not only to develop games but also to integrate them into actual mathematics lessons. In this study, GeomWiz was implemented in secondary school classrooms in Croatia and Finland, providing the context for exploring students' perceptions of DGBL compared to traditional instruction.

3. Theoretical background

3.1. Impact of DGBL on motivation and engagement in GAMMA project and GeomWiz game education

DGBL has gained increasing prominence in education, including mathematics, where students often describe the subject as abstract, difficult, or disconnected from everyday life. These perceptions contribute to low persistence, anxiety, and limited enjoyment. By embedding mathematical content into interactive environments, DGBL offers an alternative pathway for learning that can foster greater willingness to participate, reduce negative emotions, and encourage more positive attitudes toward mathematics (Hussein et al., 2022; Martinovic et al., 2014; Pan et al., 2022). Research indicates that digital games can support both affective and cognitive engagement when designed to align content with meaningful tasks. Games that integrate mathematical concepts into a coherent narrative or problem-solving structure tend to sustain intrinsic motivation more effectively than those where the content feels artificially imposed (Habgood & Ainsworth, 2011; Ku et al., 2014). Feedback mechanisms such as points, badges, and progress indicators further enhance motivation by turning repetitive practice into a more dynamic and rewarding experience.

At the same time, the effects of DGBL are not uniform. Learners respond differently depending on context, instructional integration, and personal preferences.

For example, collaborative formats can enhance enjoyment and peer support, while competitive formats may be motivating for some students but intimidating for others (Hidayat et al., 2024; Hussein et al., 2022). Moreover, when games suffer from poor usability, superficial integration of content, or excessive technical demands, they can lead to neutral or even negative effects on student engagement (Hui & Mahmud, 2023). This suggests that enjoyment alone does not guarantee educational value, and that careful attention must be paid to how game design aligns with learning objectives and cognitive processes (Ilić et al., 2024).

3.2. Understanding of motivation and engagement

Self-Determination Theory (SDT) provides a useful lens for understanding how DGBL can influence students' motivation. According to SDT, three basic psychological needs – autonomy, competence, and relatedness – must be satisfied to support intrinsic motivation (Deci & Ryan, 2004). Digital games are particularly well suited to address these needs. Autonomy can be supported through meaningful choices and flexible progression, competence through adjustable difficulty and immediate feedback, and relatedness through shared classroom experiences or engaging narratives (Liu et al., 2020; Cubillos et al., 2024). In the context of mathematics education, these features are especially relevant. Students often perceive mathematics as externally imposed and highly evaluative; by contrast, games provide opportunities to experience a sense of control and mastery without the same level of pressure. Research has shown that game elements such as scaffolding, visual rewards, and replay options can strengthen students' motivation and persistence by aligning with SDT principles (Huang et al., 2019; Proulx et al., 2017). Thus, SDT helps explain why students may find DGBL lessons more engaging and motivating than traditional mathematics instruction.

Engagement Theory emphasizes that effective technology-supported learning should be interactive, purposeful, and motivating (Kearsley & Shneiderman, 1998). Although originally designed for collaborative environments, its principles can be applied more broadly: learning activities are most engaging when they involve active participation, clear goals, and meaningful feedback (Li et al., 2024). In DGBL, these qualities are reflected in mechanics such as level progression, immediate responses to student actions, and challenges that are appropriately difficult. When students perceive tasks as purposeful and feedback as informative, they are more likely to sustain attention and enjoyment. Conversely, if the activity feels repetitive or disconnected from meaningful goals, engagement tends to diminish (Hussein et al., 2022). For mathematics education, Engagement Theory is particularly relevant because it highlights how the design of learning activities determines whether students remain motivated. This perspective helps explain why some students view DGBL as enjoyable and productive, while others may find it boring or frustrating.

Cognitive Load Theory (CLT) explains how the design of instructional materials affects learners' ability to process information. It distinguishes between intrinsic load (inherent difficulty of the material), extraneous load (cognitive effort

caused by poor design or unnecessary complexity), and germane load (mental resources directed toward meaningful learning) (Sweller, 2011). In DGBL, games can reduce intrinsic load by breaking complex problems into smaller, manageable challenges and by providing scaffolding such as hints or multiple attempts. At the same time, they may increase extraneous load if the interface is confusing, the instructions unclear, or technical issues distract from learning (Chang & Yang, 2023; Woo, 2014). Even if a game remains enjoyable, excessive extraneous load can hinder understanding and frustrate learners. For mathematics education, CLT highlights why students may respond differently to the same game-based lesson. While some benefit from simplified problem-solving and supportive feedback, others may experience overload from technical or structural limitations.

Together, SDT, Engagement Theory, and CLT provide complementary perspectives for analysing students' experiences with DGBL in comparison to their regular mathematics instruction, identifying both supportive and hindering aspects of their experiences.

4. Methodology

4.1. Context, participants and data collection

The study was conducted in spring 2023 within the European Erasmus+ project GAMMA, which brought together researchers, teacher educators, and secondary school mathematics teachers from four universities and four upper secondary schools in Greece, Croatia, the Netherlands, and Finland. As part of the project, teachers were asked to integrate GAMMA games into their regular mathematics lessons. Before implementing the games in classrooms, the teachers became familiar with the project platforms, played and modified existing game examples, and contributed to the development of GAMMA games by providing feedback on several versions during the design process.

The present study focused specifically on one particular GAMMA game, called GeomWiz and students' perceptions of lessons in which the GeomWiz game was implemented. A mixed-methods design was used, combining closed-ended questionnaire items with open-ended student responses. Data were collected in two countries where GeomWiz was used in classroom instruction: Croatia and Finland. The Croatian sample consisted of 158 students, while the Finnish sample consisted of 79 students. All participants were upper secondary school students aged between 15 and 17 years; however, exact individual age data were not collected. The classroom implementation involved seven mathematics teachers from Croatia and six mathematics teachers from Finland. Each teacher implemented the game in one classroom group, resulting in seven classroom groups in Croatia and six classroom groups in Finland. In this way, the study captured students' experiences of GeomWiz as it was embedded in regular mathematics instruction, rather than in a separate experimental setting.

In the present study, students used GeomWiz 4, GeomWiz 5, or GeomWiz 6, depending on the mathematical topic being taught in their regular mathematics

lesson. GeomWiz 4 focused on the area of a triangle. It required students to recognize different types of triangles, decide which formula was appropriate based on the given information, identify the elements needed to calculate the area, and apply the correct formula. GeomWiz 5 focused on the sine rule, requiring students to recognize when the sine rule could be applied and to use it to calculate unknown sides or angles in different triangle contexts. GeomWiz 6 focused on the cosine rule and required students to recognize when the cosine rule could be applied, calculate an unknown side or angle of a triangle, and use the rule in different triangle contexts.

The distribution of GeomWiz levels differed between the two countries. Finnish students used two levels: GeomWiz 4 and GeomWiz 6, whereas Croatian students used GeomWiz 4, GeomWiz 5, GeomWiz 6, or a combination of two levels, depending on the topic being addressed in their regular mathematics instruction. The available records are presented in Table 1. Although all students did not use the same mathematical content, the GeomWiz levels used in the study were designed within the same game environment and followed the same game mechanics, visual style, aesthetics, and gameplay dynamics. This consistency is important for the interpretation of the findings, as the study focused on students' perceptions of the DGBL experience rather than on comparing learning outcomes across different mathematical topics. In this sense, the mathematical content varied, but the structure and experience of gameplay remained comparable.

Table 1. Number of students who played GeomWiz levels.

GeomWiz level used	**Croatian students** (*N*)	**Finnish students** (*N*)
GeomWiz 4 only	26	60
GeomWiz 5 only	51	0
GeomWiz 6 only	53	19
Combination of two GeomWiz levels	28	0

Each teacher implemented one GAMMA game during a regular mathematics lesson lasting approximately 60–90 minutes. The implementation was supported by teaching scenarios developed within the project. These scenarios were intended to help teachers plan and conduct digital game-based lessons by outlining the mathematical content, prerequisite knowledge, required resources, and suggested lesson activities. Each teaching scenario included pre-game, in-game, and post-game activities, as well as optional extension tasks for students who wished to explore the topic further. The teaching scenarios for GeomWiz levels provided a shared structure for implementation across classrooms and countries. Implementation fidelity was supported through the use of common project-developed teaching scenarios and through teachers' prior involvement in the GAMMA project. However, no formal classroom observation protocol or systematic fidelity checklist was used.

After participating in the lesson, students were invited to complete a questionnaire designed to capture their perspectives (Table 2). The instrument consisted

of both closed- and open-ended questions. Closed-ended items asked students to rate their experience compared to a regular mathematics lesson (e.g., fun, motivating, engaging, confusing, frustrating) on a 5-point Likert scale. Open-ended items asked students to describe favorable and hindering aspects of the lesson. Completion of the questionnaire was voluntary, and students could skip any questions they did not wish to answer. The questionnaire was developed collaboratively by project partners and refined through several iterations to ensure clarity and relevance.

Table 2. Questions from the questionnaire.

Questions	**Type**
Compared to a regular mathematics lesson, this lesson with game was more:	
fun.	Scale*, 1–5
motivating.	Scale*, 1–5
engaging.	Scale*, 1–5
confusing.	Scale*, 1–5
frustrating.	Scale*, 1–5
What are favorable aspects of the experienced DGBL lesson?	Open
What are hindering aspects of the experienced DGBL lesson?	Open

*1 = much less, 2 = less, 3 = same, 4 = more, 5 = much more

4.2. Data analysis

Descriptive statistics were used to summarize responses to the closed-ended questions and to provide insight into overall trends in students' perceptions and experiences. Frequencies, percentages, and mean scores were calculated to examine students' responses to statements related to the implementation of DGBL. Since the five closed-ended items were designed to capture distinct dimensions of students' lesson experience rather than to form a single psychometric scale, the main analysis was conducted at the item level. Differences between countries were examined using the Mann–Whitney U test. Effect sizes were calculated as $r = z/\sqrt{N}$, where N represents the total number of participants across both groups.

Open-ended responses were analyzed using qualitative thematic coding, following the approach to qualitative data analysis described by Miles et al. (2020). The analysis focused on identifying recurring patterns in students' descriptions of favorable and hindering aspects of the implemented DGBL lessons. First, all responses were read several times to become familiar with the data. Initial codes were then developed inductively from the students' answers, following Saldaña's (2021) recommendations for open coding. In the next step, related codes were grouped into broader categories based on similarity in meaning and content. These categories were reviewed and refined through comparison across the Croatian and Finnish datasets in order to ensure coherence and consistency. During the analysis, overlapping codes and categories were merged where appropriate. For example,

comments referring to bugs, unclear guidance, platform limitations, or difficulties using the game were grouped under the category "Design and Accessibility Limitations." Similarly, responses referring to boredom, repetition, confusing tasks, or poorly structured questions were grouped under "Frustration with Game Structure or Content." The final categories were then defined and named, and the findings were presented with illustrative quotes. Students' original wording was preserved in the quotes to maintain the authenticity of their responses. Category occurrences were also counted to support comparison of patterns between the two countries.

5. Results

5.1. Students' perception of DGBL lesson

When comparing students' perceptions of experienced DGBL lessons with regular mathematics lessons, both Croatian and Finnish participants reported higher levels of enjoyment, motivation, and engagement (Figure 2).

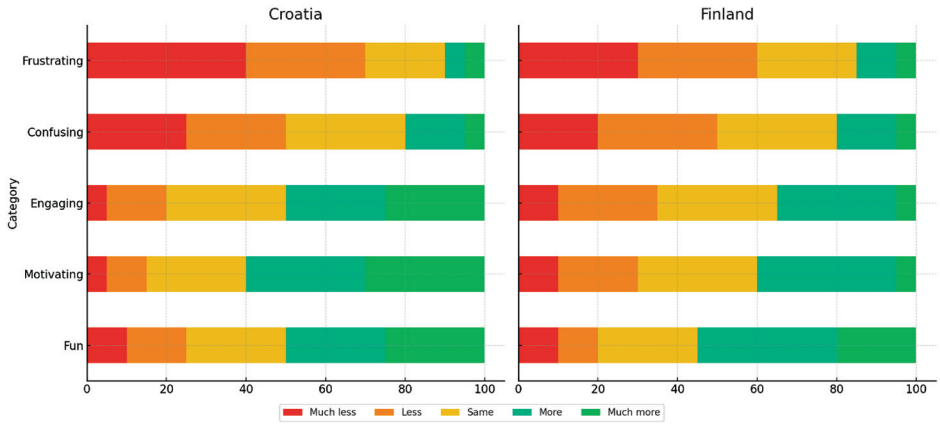


Figure 2. Comparison of Croatian and Finnish students for enacted DGBL lesson (1 = much less, 2 = less, 3 = same, 4 = more, 5= much more).

In Croatia, about half indicated that the game-based lesson was "more" or "much more fun," with similar trends observed for motivation and engagement. Finnish students also perceived the lesson as more enjoyable and motivating, though their responses were more moderate, with a greater proportion selecting "same" rather than the extreme positive categories. Similarly, when examining the negative aspects of the learning experience, no significant differences were observed between the two groups. In both the Finnish and Croatian students' responses, the majority of students reported experiencing less or much less confusion in comparison with a traditional classroom lesson. Only a small percentage of students indicated that the lesson was more or much more confusing, while the remaining students perceived no change in their level of confusion. Indicators of frustration followed a similar pattern and appeared even less pronounced. Once again, both

Croatian and Finnish students reported a clear overall reduction in frustration, with most participants describing the DGBL lesson as less or much less frustrating than a regular lesson. These findings suggest that, from the students' perspective, the introduction of game-based learning did not introduce additional negative emotional burdens; on the contrary, it tended to alleviate them.

The results for Mann–Whitney U test were reported in Table 3. There were statistically significant differences between Croatian and Finnish students for lessons being more fun and motivating. The results indicated small effects for lessons being more fun ($r = .14$) and motivating, ($r = .17$), suggesting that although the group differences were statistically significant, their practical magnitude was small. The difference for lesson being more engaging approached statistical significance but did not reach the conventional threshold of $p < .05$ and the effect size was small ($r = .12$). Regarding the negative aspects of the learning experience, no statistically significant differences were found between Croatian and Finnish students. The effect size was negligible what further suggests that the difference between the two groups was not practically meaningful. Effect sizes for negative aspects were negligible.

Table 3. Comparison of Croatian and Finnish students.

	Croatia			**Finland**					
	M (*SD*)	*Mdn*	*IQR*	*M* (*SD*)	*Mdn*	*IQR*	*U*	*z*	*p*
Fun	3.56 (1.02)	4	1	3.24 (0.98)	3	1	7250.00	2.12	.034*
Motivating	3.33 (0.99)	3	1	2.91 (1.04)	3	2	7498.50	2.62	.009*
Engaging	3.25 (1.07)	3	1	2.96 (0.94)	3	2	7129.00	1.86	.063
Confusing	2.37 (1.02)	2	2	2.37 (1.02)	2	1	6061.50	−0.37	.710
Frustrating	2.13 (0.89)	2	1	2.29 (1.07)	2	2	5548.00	−1.45	.147

Note. *Mdn* = Median, *IQR* = Interquartile Range. *$p < .05$

5.2. Favorable aspects of DGBL lesson

This open question was answered by 57 Croatian students and 47 Finnish students. The analysis of students' reflections on the DGBL lesson identified three main categories of favorable aspects: Positive Affective Experience, Better Understanding and Retention, and Novelty and Digital Format (Table 4). Because a single response could contain more than one idea, the results are reported as coded occurrences rather than as numbers of students. This approach provides a clearer indication of how often each category appeared across the two datasets.

Themes are reported using the number of coded occurrences rather than the number of students. Occurrence counts provide a more accurate representation of the distribution of themes.

Positive Affective Experience: The most dominant theme across both Croatia and Finland was students' perception that the digital game lesson was fun, engaging, and motivating (Table 4). Students explicitly described the game as enjoyable, less boring, or as increasing their willingness to participate compared to traditional classroom activities. Students frequently associated the playful format with stronger focus, comfort, and interest. Croatian participants emphasized that the activity was "more relaxing than regular study" and that "you have fun while you learn." Finnish students were similarly enthusiastic, describing the lesson as "way more fun than usual," "better than normal math lessons," and noting that "the game makes focusing a bit easier." In Finland, several students also emphasized the emotional impact of game elements like music and character design, which contributed to their sense of engagement. This suggests that affective engagement, rooted in enjoyment and motivation, was a core educational benefit in both countries.

Better Understanding and Retention: A second prominent theme related to the perceived learning value of the digital game (Table 4). Students pointed out that the lesson format supported faster learning, easier comprehension, or better retention. Croatian students commonly framed this theme in terms of efficiency and productive practice, expressing that the game enabled them to "fast learn a lesson," was "good for practice," and allowed them to "learn more in less time." Finnish students, meanwhile, highlighted improved comprehension and help with challenging material, citing that "it makes learning difficult things more fun" and "studying on your own made understanding easier." Multiple Finnish responses mentioned the value of game mechanics such as multiple attempts and visual aids in supporting their learning. This indicates that while both groups valued the learning benefits, Croatian students conceptualized these benefits as productivity and practical gains, whereas Finnish students highlighted the game's role in supporting conceptual understanding and deeper mastery.

Novelty and Digital Format: The third theme related to appreciation of the digital and game-based format itself (Table 4). Students' reasons varied by context. Croatian participants stressed the practical advantages of digitalization, such as the ability to "study anywhere since it's on the phone," the fact that the game required "less writing," and the perception that it was a "more engaging" alternative to textbooks. However, in Finland novelty was discussed in terms of aesthetics, design, and game feel. Students mentioned the "cute character," "good music," and praised the "horizontal joystick design" for making the lesson feel like a real game rather than simply an educational exercise. Others emphasized that the experience was a "good change from usually boring math books" and valued the "different environment." This contrast suggests that while Croatian students emphasized functional convenience, the Finnish students drew attention to visual, sensory, and design-related features as central to their enjoyment.

Mixed, Minimal, or Unclear Responses: Some responses were too brief, vague, or unrelated to be interpreted reliably. These included one-word answers such as

"yes" or "good," symbols such as "." or "-", and humorous or off-topic remarks. Because these responses did not provide enough context to identify a clear favorable aspect of the DGBL lesson, they were treated as non-codable and reported separately in Table 4.

Table 4. Favorable aspects of GeomWiz game.

Theme	**Finland***	**Croatia***	**Illustrative Quotes**
Positive Affective Experience	36	39	"It was fun and made this really comfortable." (Finland); "You are learning while playing." (Croatia); "Everyone likes playing games..." (Croatia)
Better Understanding and Retention	17	18	"You can learn sine rule very easy with this game." (Croatia); "Game is good because I can fast learn lesson." (Croatia); "This kind of lesson is a lot easier." (Finland)
Novelty and Digital Format	12	15	"This was a good change from the usual stuff." (Finland); "The illusion of freedom that is created by using phones in class." (Croatia)
Mixed, Minimal, or Unclear Responses	14	10	"Yes", ".", "Idk", "yas"

* number of occurrences of a particular theme

5.3. Hindering aspects of DGBL lesson

This open question was answered by 81 Croatian students and 42 Finnish students. Analysis of students' reflections on the hindering aspects of the DGBL mathematics lesson identified three main categories of hindrances (Table 5): (1) design and accessibility limitations, (2) frustration with game structure or content, and (3) no negative aspects reported. Responses from both Croatia and Finland were thematically coded using a multiple coding approach, so individual comments could be assigned to more than one category when overlapping issues were present. Themes are reported using the number of coded occurrences rather than the number of students. Occurrence counts provide a more accurate representation of the distribution of themes.

Design and Accessibility Limitations: The most prevalent theme across both Croatia and Finland reflected students' concerns with technical barriers and usability issues (Table 5). Students described problems including slow performance, game lag, device incompatibility, poor instructions, lack of explanations for wrong answers, and difficulties obtaining help. Croatian participants especially emphasized basic technical challenges. Several also noted that the use of mobile phones made it easy to get distracted or cheat. Finnish students frequently raised issues around screen size, lack of feedback, battery problems, and insufficient guidance. In both countries, these limitations contributed to students' frustration and made the game harder to use or less effective as an educational tool.

Frustration with Game Structure or Content: This theme captured students' dissatisfaction with the game's design, depth, and overall engagement value (Table 5). Students reported that the lesson was boring, too repetitive, lacked a sense of progression, or did not resemble a true game experience. Some felt that the activity functioned simply as a quiz, citing missing features such as narrative, goals, or rewards. Others reported confusing question structures, including duplicated answers or mistakes in exercises. Students' responses collectively indicate that for many the DGBL lesson failed to meet expectations for what makes games both enjoyable and challenging.

No Negative Aspects Reported: A substantial portion of students in both groups explicitly stated that they had no negative experiences with the DGBL lesson (Table 5). Students reported no complaints, using phrases such as "None," "No negative aspects," "No, everything is awesome," and "There are none." This indicates that, despite reported hindrances, a large proportion of students in both countries found the lesson acceptable or satisfactory, or at least did not feel strongly about any particular shortcomings.

Table 5. Hindering aspects of DGBL lesson.

Theme	**Finland***	**Croatia***	**Illustrative Quotes**
Design and Accessibility Limitations	20	27	"The game does not work on Galaxy A6." (Croatia); "No explanation of how to play." (Croatia); "My screen was too small." (Finland)
Frustration with Game Structure or Content	24	21	"It's just a quiz with extra steps." (Croatia); "There were two same answers but only one was accepted." (Croatia); "No storyline." (Finland)
No Negative Aspects Reported	17	33	"None."; "No negative aspects."; "Everything is awesome."

* number of occurrences of a particular theme

6. Discussion

The findings of this study provide valuable insights into how DGBL, specifically through the GeomWiz game, is perceived by secondary school students in Croatia and Finland. Overall, the results suggest that DGBL can serve as a meaningful alternative to traditional mathematics instruction, particularly in terms of enhancing student motivation, engagement, and enjoyment. Below we discuss our findings in more detail.

6.1. Favorable aspects of DGBL lessons

Students in both Croatia and Finland emphasized enjoyment, motivation, and increased willingness to participate as central benefits of the GeomWiz lesson. These findings are consistent with previous research showing that DGBL can transform

mathematics learning into a more engaging and less anxiety-inducing experience (Hussein et al., 2022; Fadda et al., 2022). In particular, the positive affective responses observed here support earlier results that students perceive digital games as "more fun" and "more motivating" than traditional classroom activities (Martinovic et al., 2014; Pan et al., 2022).

From the perspective of SDT, the design of GeomWiz supported students' psychological needs for autonomy and competence. Autonomy was facilitated by features such as the option to choose levels and proceed at an individual pace, while competence was reinforced through immediate feedback, multiple attempts, and rewards for successful problem solving. Previous studies have shown that such design elements are strongly associated with increased persistence and intrinsic motivation in digital learning environments (Liu et al., 2020; Proulx et al., 2017). Although the game lacked a multiplayer mode, the shared classroom context still created a sense of relatedness, aligning with findings that even single-player digital games can foster social connections when embedded in group learning settings (Cubillos et al., 2024).

The students' reflections also resonate with Engagement Theory. Students valued the novelty of the digital format and the purposeful challenge embedded in the game structure. Similar to earlier findings (Habgood & Ainsworth, 2011; Li et al., 2024), the integration of clear goals and immediate feedback transformed practice tasks into a more dynamic experience, reducing the perception of mathematics as repetitive.

6.2. Hindering aspects of DGBL lessons

Despite the generally positive responses, students in both countries reported technical and design-related frustrations that limited the value of the GeomWiz lesson. Technical barriers such as lagging performance, small screen size, and unclear instructions represent sources of extraneous cognitive load. According to CLT, such distractions divert mental resources away from meaningful learning and reduce instructional efficiency. Previous research similarly found that usability problems, platform constraints, and poorly explained tasks diminish the effectiveness of DGBL, even when the games are otherwise engaging (Hui & Mahmud, 2023; Chang & Yang, 2023). Students also criticized aspects of the game's structure, including repetitive tasks, lack of progression, and minimal narrative. These weaknesses undermine the sense of purpose and novelty that Engagement Theory identifies as central to sustaining motivation (Kearsley & Shneiderman, 1998). Similar criticisms have appeared in prior research, where learners reported that poorly designed educational games often resemble conventional quizzes rather than immersive game experiences (El Mawas et al., 2022; Hussein et al., 2022).

6.3. Cross-country differences

Although students in both countries generally responded positively to the GeomWiz lesson, their open-ended comments pointed to somewhat different emphases. Croa-

tian students more often described the lesson in terms of efficiency, practice, and learning more quickly. Finnish students, on the other hand, more often referred to understanding, independent work, and support in dealing with difficult content. These differences suggest that students may have evaluated the game-based lesson in relation to what they usually expect from mathematics learning. This interpretation should be read as tentative. The study did not include direct measures of classroom practice, curriculum implementation, teacher beliefs, or students' previous experiences with mathematics instruction. Therefore, the observed differences cannot be interpreted as direct evidence of national pedagogical cultures. Rather, they indicate possible contextual patterns that would need to be examined more systematically in future research.

Previous studies support the relevance of such contextual interpretations. El Mawas et al. (2022), for example, found that the same STEM game produced different motivational effects in different European settings, despite comparable gains in subject knowledge. Similarly, Nalipay et al. (2020) showed that the role of competence support may vary across cultural contexts. These findings suggest that students' sense of what makes a digital learning activity useful, motivating, or meaningful may depend partly on the educational context in which the activity is experienced. In the present study, the Croatian emphasis on efficient practice and the Finnish emphasis on understanding and independent work should be viewed as a possible pattern instead of a firm conclusion about national differences. The results suggest that the same digital game may be experienced differently depending on the classroom and curricular context in which it is used. Future studies should explore this issue by combining student perception data with classroom observations, teacher interviews, curriculum analysis, and information about students' previous experience with digital games and mathematics learning.

7. Conclusion

The findings of this study suggest that the implemented DGBL lesson was generally perceived positively by secondary school students in both Croatia and Finland. Students tended to describe the lesson as more enjoyable, motivating, and engaging than a regular mathematics lesson, while also reporting lower levels of confusion and frustration. These results indicate that, from the students' perspective, the use of GeomWiz provided a more positive learning experience than traditional classroom instruction. At the same time, the findings show that the value of DGBL does not depend only on the presence of game elements, but also on the quality of their design and classroom implementation. Features such as clear goals, meaningful feedback, appropriate challenge, and opportunities for autonomous work appear important for supporting students' engagement. This is consistent with previous research emphasizing the role of scaffolding and well-designed support in digital game-based learning (Chang & Yang, 2023). However, technical difficulties, unclear instructions, repetitive tasks, and limited game-like features may reduce the educational value of the lesson and contribute to frustration.

The comparison between Croatian and Finnish students suggests some differences in how students responded to the DGBL lesson, particularly in relation to enjoyment and motivation. However, these differences should be interpreted cautiously. They point to possible contextual patterns rather than firm conclusions about national or cultural differences. The study indicates that DGBL can be a promising addition to secondary mathematics teaching when it is carefully designed, clearly integrated into the lesson, and aligned with students' learning needs.

7.1. Limitations and further research

Several limitations should be noted when interpreting these findings. First, the study relies on self-reported perceptions, which provide insight into students' experiences but not direct evidence of learning outcomes. Future research could complement perception data with performance measures. Second, although the same game environment was used, students did not all work with the same GeomWiz level or mathematical content. This should be taken into account when comparing responses across countries, as differences in topic difficulty or classroom context may have influenced students' perceptions. Future studies could examine the same game level and mathematical topic across different contexts in order to strengthen comparability. Furthermore, the study was conducted in only two countries, which limits the generalizability of the cross-cultural comparisons. The observed differences between Croatian and Finnish students should therefore be interpreted as possible contextual patterns rather than as broad cultural conclusions. Expanding the study to more countries and classroom settings would provide a more nuanced understanding of how educational context shapes students' responses to DGBL. Finally, the study focused primarily on students' perspectives. Including teachers' perspectives would provide a fuller understanding of the opportunities and challenges involved in integrating digital games into secondary mathematics teaching. Future research should therefore adopt a mixed-perspective approach, combining student data, teacher data, classroom observations, and learning outcome measures.

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Contact addresses:

Ljerka Jukić Matić
School of Applied Mathematics and Informatics
Josip Juraj Strossmayer University of Osijek
Trg Ljudevita Gaja 6, 31 000 Osijek, Croatia
e-mail: ljukic@mathos.hr

Darija Marković
Faculty of Civil Engineering and Architecture Osijek
Josip Juraj Strossmayer University of Osijek
Vladimira Preloga 3, 31 000 Osijek, Croatia
e-mail: dmarkovic1@gfos.hr

Stavovi učenika o igri GeomWiz: Analiza učenja matematike temeljenog na digitalnim igrama u srednjim školama

Ljerka Jukić Matić[1] i Darija Marković[2]

[1]Fakultet primijenjene matematike i informatike, Sveučilište Josipa Jurja Strossmayera u Osijeku, Hrvatska
[2]Građevinski i arhitektonski fakultet Osijek, Sveučilište Josipa Jurja Strossmayera u Osijeku, Hrvatska

Sažetak. Učenje temeljeno na digitalnim igrama sve se više prepoznaje kao učinkovit pristup poučavanju matematike, jer pozitivno utječe na školski uspjeh učenika, njihovu motivaciju, sposobnost rješavanja problema i samopouzdanje. Iako je takav pristup široko primijenjen u osnovnim školama, u srednjoškolskom obrazovanju još je uvijek slabo zastupljen. Jedan od razloga za to je upravo nedostatak digitalnih igara posebno osmišljenih za poučavanje apstraktnih i složenijih matematičkih pojmova koji se uvode u srednjoj školi. Kako bi se taj nedostatak ublažio, u sklopu Erasmus+ projekta GAMMA razvijene su digitalne igre i pripadajući nastavni scenariji, s naglaskom na teme iz srednjoškolske matematike, poput geometrije. Cilj ovog istraživanja bio je ispitati stavove učenika o igri GeomWiz te njihova iskustva s nastavnim aktivnostima koje ju prate. Podaci su prikupljeni pomoću upitnika koji je sadržavao kombinaciju zatvorenih i otvorenih pitanja, a proveden je među učenicima u Finskoj i Hrvatskoj. Za analizu odgovora na zatvorena pitanja korištena je deskriptivna statistika, dok su otvoreni odgovori analizirani kvalitativno. Rezultati istraživanja pružaju vrijedne uvide u angažman učenika, njihovu percepciju učenja uz pomoć digitalnih igara te potencijal ovakvih alata za obogaćivanje nastave matematike u srednjoj školi.

Ključne riječi: učenje temeljeno na digitalnim igrama, digitalne igre, matematika u srednjoj školi, igra GeomWiz

Minecraft as a Mathematical Playground: A Review of Educational Applications, Computational Thinking, and Mathematical Insights

Ana Mirković Moguš* and Željko Gregorović

Faculty of Education, Josip Juraj Strossmayer University of Osijek, Croatia

Abstract. Game-based learning (GBL) is increasingly recognized for its ability to enhance student engagement and conceptual understanding. Among digital GBL tools, Minecraft Education is a prominent digital tool that encourages creativity, collaboration, and problem-solving which are essential skills for learners. In mathematics education, where students often face challenges with abstract concepts and may lack motivation in traditional settings, Minecraft provides a dynamic 3D environment where learners can engage with mathematical ideas through spatial reasoning, geometric explorations, measurement, and pattern recognition. However, there is limited research on Minecraft's educational effectiveness, cognitive impact, and alignment with curriculum standards. The aim of this paper is to review the growing body of research at the intersection of Minecraft, mathematics and computer science. The analysis was conducted by following the PRISMA guidelines and used a narrative synthesis with a qualitative content analysis to arrange results from mathematics, computer science education, and motivation outcomes. Results show that Minecraft can facilitate a range of educational results, including creativity, teamwork, critical thinking, motivation, active learning and mastery of content. The review also identifies gaps in empirical evidence and alignment with curriculum standards. The findings contribute to better understanding how valuable Minecraft can be in teaching and learning. It also opens up possibilities for future studies in this growing area that connects different fields.

Keywords: game-based learning (GBL), Minecraft Education, mathematics education, computer science education, student engagement

*Corresponding author.

1. Introduction

The continuous growth of digital games for entertainment among children, teenagers, and young adults has led to exploration of their potential in education. Although there is increasing interest in digital game-based learning (DGBL) (Squire, 2022), its effects on learning and cognition differ depending on the specific digital games used (Barz et al., 2023). Digital games designed with educational purposes in mind improve learning experiences by addressing particular educational needs (Kucher, 2021). Minecraft, developed by Mojang Studios and launched in 2011, is a sandbox video game that has become incredibly popular and influential. As of July 2025, the game has more than 204.33 million monthly active players, which is a clear indication of the game's sustained global popularity and massive player engagement (Priori Data, 2025). The number of people playing Minecraft live on the internet, according to the latest data, changes during the peak hours of gaming and the daily Minecraft live player count is according to 2.2 and 3.2 million users who are online at the same time, while number of daily users counts on average 31.96 million players (PlayerCounter). The United States is at the top of the list when it comes to the number of Minecraft players by country, having about 21.21% of the daily users. Next spots are related to the countries of Europe and Asia-Pacific regions. The US itself accounts for about 43.3 million registered players, hence making it the biggest one single market for Minecraft in the world. The demographic segment of children and teenagers between the ages of 15 and 21 years constitute 43% of the total user base of Minecraft, thus making them the largest single group of the said users. There are 54% of the Minecraft players who are males, and 32% who are females out of which 14% have not been specified. The game attracts diverse audiences across all age groups. Minecraft, which is the favourite game for building in 3D, has become the most preferred tool to enhance different learning domains that is spread over primary, secondary and higher education (AlJanah et al., 2023; Andersen & Rustad, 2022). This review is based on the constructivist learning theory, which views learning as an active process of knowledge construction through exploration, problem-solving, collaboration, and reflection. In this regard, Minecraft is highly appropriate as it offers learners the possibility to create, test, alter, and share ideas in a transparent and engaging setting. However, most of the current studies on the educational use of Minecraft are scattered. Research typically focuses on either the effects of the game on math skills, such as spatial reasoning and geometry, or on computer-science skills, such as coding, logic, and computational thinking. There are only a few studies that explore the overlap of these areas in actual practice. This creates a gap in educational research as it remains unclear under which instructional conditions Minecraft supports meaningful learning apart from just motivation and engagement. Therefore, a targeted review of the relation between math and computer science education would be very useful, as many Minecraft based activities are by their very nature combining measurement, geometry, logic, algorithms, problem solving, and design-based learning. Minecraft's popularity and extensive use in educational settings implies the need to find out the supporting evidence for its use. This study provides a comprehensive review of the evidence supporting the efficacy of Minecraft in improving learning outcomes in mathematics and computer science education.

2. Methodology

This systematic review has been documented according to the criteria set by the PRISMA (Preferred Reporting Items for Systematic Reviews and Meta-Analyses) Statement (Page et al., 2021). The initial search across three databases (Scopus, Web of Science, ERIC) was conducted on December 15, 2024. Part of the search strategy was to find articles with Minecraft-related terms in their titles and abstracts. The sources used in this research include scientific academic papers, which were retrieved from the relevant databases before mentioned. The databases were searched using a combination of keywords and phrases and the search operator AND ("Minecraft" AND "education", "Minecraft" AND "mathematics education" and "Minecraft" AND "computer science education"). Keywords and phrases were searched in the titles of scientific papers, abstracts and keywords provided by the authors of the papers. Additional secondary search was also performed through Google Scholar and Open Grey. Database searching was repeated in March and April 2025 to update the original search. The findings were made more valid by using a priori inclusion and exclusion criteria to filter out only those studies that were relevant to research questions. The search process identified 57 records: $n = 23$ from Scopus, $n = 17$ from Web of Science, $n = 12$ from ERIC, and $n = 5$ from supplementary sources. Because several records are indexed in more than one database, these numbers should be interpreted as approximate database-coverage counts. After removing 3 duplicates, 54 records were screened by title and abstract. Of these, 9 records were excluded because they did not meet the inclusion criteria. 45 full-text papers were then assessed for eligibility, and 14 were excluded with reasons, including: not focusing on Minecraft as the main educational tool, not addressing mathematics or computer-science learning outcomes, involving adult participants, being single-case studies, or not being written in English. Inclusion criteria were related to papers written in English, Minecraft used as a main tool, studies that investigated Minecraft involving at least one learning outcome in mathematics or informatics education. Screening was conducted by two reviewers. Each reviewer independently screened titles, abstracts, and full texts, and disagreements were resolved through discussion until consensus was reached. The quality appraisal was used to support interpretation of the evidence rather than to exclude studies from the review. The synthesis of the included studies was done by a narrative synthesis supported by qualitative content analysis. From each study, information was collected on the educational domain, participant group, Minecraft-based activity, learning outcomes, research design, and main findings. Initially, the results were structured deductively following the three review questions. Subsequently, they were categorized into common synthesis themes after several readings of the studies. Literature review followed these questions:

1. How effective is Minecraft as a tool for teaching spatial reasoning and geometric concepts compared to traditional methods?
2. How can Minecraft's Redstone mechanics be used to teach Boolean logic, circuitry, and computational thinking?
3. What role does Minecraft play in fostering motivational and engagement outcomes among students?

3. Minecraft in mathematics education

Minecraft has been acclaimed by many as an open-ended game, which makes it ideal for unleashing one's creativity and adaptable to any need. The game has been associated as a virtual playground or a sandbox for education (Karsenti & Bugmann, 2017). In addition, Minecraft has been used as a tool to foster learners' motivation, engagement and interest which is beneficial for learning (Short, 2012). Students were highly motivated when they were involved in an open-world setting which was methodically connected to the mathematics and computer science curricula (Ivanov & Yordanov, 2024). Minecraft can enhance learning outcomes in mathematics by helping students develop their numeracy skills, such as arithmetic, geometry, and measurement. Students can use Minecraft to experiment with math concepts, find solutions to problems, and practice their learning in real-life situations. Several studies have suggested that playing Minecraft can increase students' understanding of various subjects, including mathematics (Jensen & Hanghøj, 2020; Meier et al., 2020; Carbonell-Carrera et al., 2021; Tangkui & Keong, 2021). The block-based environment of Minecraft can be utilized to introduce mathematical concepts (Jensen & Hanghøj, 2020). Minecraft environment is built on blocks, each one symbolizing a cubic meter in the game's scale. Such a basic setup is a great way to introduce or explain math concepts, especially aspects of geometry and spatial reasoning (Worsley & Bar-El, 2020). Hence, teachers may have students build and change geometric structures, thereby deepening their understanding of geometry through hands-on experience (Kervin et al., 2016). In the study by Jensen & Hanghøj (2020), the use of mathematics and geometry applications was found to be highly effective, especially for spatial reasoning and geometric concept visualization. These findings are also supported by the research of Ivanov and Yordanov (2024) emphasizing the block-grid environment of Minecraft as a suitable visual representation for mathematics tasks such as coordinate geometry or measurement. The research of Carbonell-Carrera et al. (2021) was concerned with the effects of a block-building workshop, which used Minecraft activities, on the development and training of students' mental rotation skill. Participants after the workshop demonstrated an increase in performance on the mental rotation task. McCashin et al. (2019) researched changes in students' understanding of geometry after attending a five-day Minecraft camp, but no improvements were found. Similarly, the results of Hughes et al. (2020) found no statistically significant improvement in students' spatial reasoning. The games flow of building process brings into focus the need for calculating volumes, areas, and perimeters, thereby improving the learners' spatial thinking skills. Such a virtual activity is a replica of the real one and serves as a stimulating and natural way of math learning. Grobelna et al. (2022) investigated the use of Minecraft to teach mathematical modeling Petri nets to undergraduate students. The results demonstrate that instructing interpreted Petri nets within Minecraft based settings may yield better results in the final evaluations and make the class more interesting and engaging.

Furthermore, numbers and counting are also dominant in using Minecraft. For example, the gathering and utilization of materials is a process that requires the knowledge of quantity and simple arithmetic. While making things, students have to figure out how many materials they have, how many are necessary, and how many will be free after. Minecraft also opens up numerous fraction learning

possibilities. Fractions and fraction operations are one of the most basic yet most difficult to understand concepts in math for many students. Activities could use ratio, fractions and proportions (Meier et al., 2020). Research shows that students' performance in solving fractions improved after finishing fraction activities in different Minecraft worlds (Tangkui and Keong, 2021). For instance, students may create building according to given proportions, or perform operations that require the even distribution of resources among a number of characters or places. The virtual world is made up of blocks which students can utilize to create a visual representation of fractions. They may consider a full block as one whole unit, and then employ smaller blocks or divided sections of a block to show fractional parts. Through these interactive activities, students get direct, experimental contact with number concepts. Students get a chance to realize the real-world applications of fractions, thus it becomes an original learning experience which can be very effective in improving their mathematical literacy.

4. Minecraft in computer science education

Minecraft is an open-ended, sandbox game whose design is in line with the constructivist learning theory. This means that learners have the freedom to create artifacts and test their ideas by iterative making. In the context of computer science education, this is reflected through the concept of learning by design wherein students demonstrate their understanding of algorithms by creating mechanisms that are part of the world (e.g., redstone circuits) or writing code that alters world state. The platform allows very clear and fast feedback (game loop, visible state changes), low floor/high ceiling (transition from playing to formal system), social construction (working together on shared worlds) and abstractions (redstone as logic gates). Minecraft's Redstone mechanics serve as an engaging platform for teaching Boolean logic, circuitry, and computational thinking. By utilizing the game's interactive environment, educators can create hands-on experiences that enhance understanding of these concepts. Using Minecraft is beneficial for supporting and nurturing abilities as collaboration, communication, problem-solving, critical thinking, creativity and digital literacy (Nebel et al., 2016). The programming education area is shown by Kane et al. (2019) who came up with a Minecraft-based game to make the first steps in Python appealing to young learners. Prayaga et al. (2016) showed how students successfully and creatively finished task of designing their own game in Minecraft, which included one or two logic gates. Hewett et al. (2020) study how contemporary skills can be obtained via the design of video games.

Several studies investigated the influence of Minecraft on computational thinking. Improvements in children's computational thinking skills are reported in the paper of Koivisto et al. (2019) where authors explored the impact of Earth 2.0 Minecraft (an augmented reality game based on Minecraft) implemented as part of an afterschool program. Kutay and Oner (2022) assessed the effects of a Minecraft Education coding activities on students' computational thinking skills and the results also showed scores improved significantly. Qu et al. (2023) also showed a positive effect on computational thinking by examining whether a lesson on Python programming in Minecraft influenced on high-school students computational thinking skills.

Minecraft allows students to construct logical gates (AND, OR, NOT) using Redstone, making abstract concepts tangible (Bermudes-Mora & Naranjo-Sánchez, 2024). The use of game mechanics fosters a competitive and collaborative atmosphere, enhancing student engagement and motivation (Bermudes-Mora & Naranjo-Sánchez, 2024). Students can experiment with digital circuits in a safe environment before transitioning to real-world applications, reinforcing their understanding of circuit design (Faust, 2024). Building circuits in Minecraft encourages critical thinking and troubleshooting, essential skills in electronics (Faust, 2024). Minecraft's Redstone mechanics simulate digital circuits, allowing students to construct logical gates and circuits visually. This hands-on approach enhances understanding of Boolean logic and circuitry, fostering computational thinking through experimentation and problem-solving in a familiar, engaging environment (Faust, 2024). The game also offers a gradual introduction to programming concepts, allowing students to start with basic block building activities and then move on to more complicated coding tasks. Furthermore, group projects that foster collaboration give students the opportunity to co-animate, and share different learning strategies among them, which can enhance students' computational literacy (Mørch & Andersen, 2023).

5. Motivational and engagement outcomes

Several studies researched motivational and engagement outcomes of using Minecraft in education. Immersive and interactive feature of Minecraft seems to have been one of the major reasons why students got interested and were consequently involved in learning process. In a study by Opmeer et al. (2019), they examined the role of a 16-week sustainable spatial planning project in Minecraft on students' emotional engagement. The group which engaged in the activity through Minecraft showed a significantly higher affected emotions score. De Sena & Jordão (2021) observed that the students were very much interested and concentrated in the performance of their Minecraft-based cartography projects. The results of McColgan et al. (2018) showed significant improvements in students' perceptions of their skills using technology and confidence with new technology where students completed a Minecraft project which consisted of playing Minecraft, participating in sample physics lessons, developing their own Minecraft environment, forming assessments and lesson plans. Sulaiman et al. (2024) wrote about raised engagement and motivation of students when learning solar photovoltaic technology with the help of Minecraft. Tablatin et al. (2023) examined the effects of students participating in What-If-Hypothetical implementations in Minecraft on their STEM interest. Although they did not find statistically significant changes for overall score on STEM interest questionnaire, they did find statistically significant improvement on the Choice Actions subscale of the questionnaire (which measures perceptions that actions related to STEM will assist with a future career). According to a study conducted by Samsudin et al. (2022) students demonstrated improved learning skills following a 10-week intervention using Minecraft Education. As part of the study, students used the platform to explore, construct, and recreate worlds using specific topics (e.g. create a home using renewable energy, as sustainability). Overall, the outcome measure increased from pre- to post- intervention. However, it is still a point of interest that engagement by itself does not seem to be a factor that would

automatically lead to better learning outcomes. In a number of cases, the research results have shown that student motivation has been positively affected, whereas, the proof for cognitive gains has been less certain. For example, Slattery et al. (2024) did not find any significant changes in spatial thinking in whole group, but subgroup analysis showed that younger students benefited.

6. Results

The three synthesis categories are presented in Table 1 along with the key findings for each category and the number of studies that have contributed to each category. The categories overlap, as some of the studies covered more than one educational outcome.

Table 1. Evidence supported answers to the research questions.

Research question	**Results**	**N***	**References**
How effective is Minecraft as a tool for teaching spatial reasoning and geometric concepts compared to traditional methods?	engagement and motivation hands-on, visual learning real-world application scaffolded learning	15	Ivanov & Yordanov (2024) Slattery et al. (2024) Grobelina et al. (2022) Carbonell-Carrera et al. (2021) Jensen & Hanghøj (2020) Meier et al. (2020) Tangkui & Keong (2021) Hughes et al. (2020) Worsley & Bar-El (2020) McCashin et al. (2019) Karsenti & Bugmann (2017) Foerster (2017) Nebel et al. (2016) Kervin et al. (2016) Short (2012)
How can Minecraft's Redstone mechanics be used to teach Boolean logic, circuitry, and computational thinking?	interactive learning gamification virtual simulations problem solving skills stepwise learning teamwork communication peer learning	10	Nebel et al. (2016) Bermudes-Mora & Naranjo-Sánchez (2024) Faust (2024) Mørch & Andersen (2023) Hewett et al. (2020) Kane et al. (2019) Koivisto et al. (2019) Kutay & Oner (2022) Prayaga et al. (2016) Qu et al. (2023)
What role does Minecraft play in fostering motivational and engagement outcomes among students?	higher affected emotions increased interest increased concentration increased self-confidence raised engagement raised motivation improved learning skills enjoying the maze challenges	8	De Sena & Jordão (2021) Sulaiman et al. (2024) Opmeer et al. (2019) McColgan et al. (2018) Sulaiman et al. (2024) Tablatin et al. (2023) Samsudin et al. (2022) Hughes et al. (2020)

* number of contributing papers

Synthesis of the results yielded a conceptual framework which shows intersection of Minecraft, mathematics and computer science education (Figure 1).

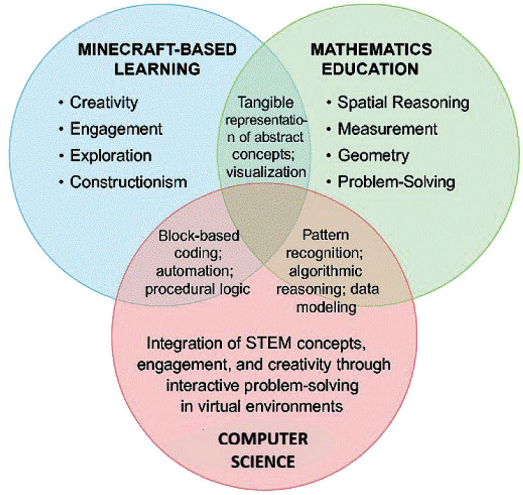

Figure 1. Conceptual framework (intersection of Minecraft, mathematics and computer science).

7. Discussion

The results shown by this review indicate that Minecraft is widely acknowledged as an effective educational tool to enhance students' motivation, engagement, collaboration, creativity, and active participation. Especially within the realm of mathematics and computer science teaching, the convincing evidence has come from the use of Minecraft not just as a game but as a well-structured educational environment where students construct, experiment, refine, and articulate their ideas. Several studies have shown that spatial reasoning, geometric visualization skills, and computational thinking improve in students if the tasks given are well-supported and are in line with the set objectives. Nevertheless, the impact on students' performance in the case of short, loosely organized, and broadly measured implementations that are not closely related to the in-game activities is somewhat mixed. Some studies show that the effects are null or heterogeneous when the exposure is brief and when the assessments are targeting skills beyond those in-game tasks (e.g. standardized spatial tests after a short intervention). This pattern suggests that the two factors, i.e. the fidelity of implementation and the task assessment alignment, being the main moderators of the impact.

Considering the first research question interpretation of the findings are following. In different reports, students use Minecraft's block-grid and 3D world to bring geometric concepts to the level of their reality, change constructions, and take measurements all of which provide them with tangible ways of accessing abstract content. The positive effects are mainly associated with students physically building models, being engaged in visualization, and performing tasks that develop spatial thinking, all of which were tasks that had been specifically designed with clear objectives and feedback. Overall, studies confirm results of Foerster (2017)

that Minecraft is an effective math educational tool which can be conveniently employed to enhance the present curriculum.

The second research question yields further findings. Redstone circuits allow Boolean logic to be something that can be physically interacted with, thus making the concept more understandable for the learners as they go through the process of designing, testing, and debugging the artifacts. The most significant improvements in understanding have been reported when the teaching method is an explicit "learning by design" approach: students construct gates or mechanics that interact with the world, get instant feedback, and gradually move from the physical builds to formal concepts. Collaborative builds seem to introduce the social scaffolding which fosters metacognitive talk. These situations are in line with constructivist theory and probably are the main reasons for the significant positive changes in computational thinking that have been documented in a number of implementations.

Related to the third research question evidence repeatedly demonstrates that Minecraft improves the students' interest, affect, and on-task behavior and often their self-confidence when using technology. It is worth noting that, higher engagement, by itself, does not necessarily lead to a gain in achievements, instead, cognitive outcomes are the result of factors such as the motivational lift, clear learning goals, structured challenges, and formative feedback. First, cognitive misconceptions are clarified, correcting the common incorrect view that Minecraft inherently produces direct educational effects. It is pointed out that its educational value depends entirely on how it is used, and that it serves as a core teaching tool in educational settings. Frontline teachers must plan in advance when designing courses and define clear learning objectives. For mathematics classes, teachers must assign tasks that require the application of measurement, geometry, proportional reasoning, and spatial visualization skills, and must not only arrange unstructured free building. Implementation guidelines for integrated computer science teaching based on Minecraft can be proposed. First, link Redstone, in-game programming tasks, and scenario-based problem-solving to core computer science concepts including algorithms, logic gates, sorting, debugging, and abstraction. Second, require teachers to support reflection, discussion, and formative feedback, and build connections between in-game behaviors and subject-matter knowledge and third, clearly define roles, plans, and assessment criteria for collaborative activities in advance, to prevent Minecraft from being used only as a fun-based motivational tool. This review has three explicit limitations. First, its literature search only covered specified databases and included only English-language studies, which may have omitted relevant non-English research or studies with limited access channels. Second, this review focused exclusively on the single game Minecraft, so its conclusions cannot be generalized to all types of game-based learning scenarios. Third, the included primary studies differ substantially across dimensions such as research design and sample size, meaning they cannot support conclusions with strong generalizability.

Cumulative synthesized evidence from existing research shows that Minecraft can serve as an effective constructivist learning environment to support mathematics and computer science education, enabling learners to convert abstract concepts

into tangible, interactive, and shareable concrete content within learning communities; this review identifies conceptual overlaps between the two disciplines and the tool, which allows to pinpoint application scenarios that deliver the strongest educational benefits and have the highest replicability. As such, future research should not limit itself to only verifying the tool's effectiveness, but instead focus on how instructional design, implementation duration, learning scaffolds, and assessment methods lead to meaningful learning outcomes.

8. Conclusion

Existing research indicates that Minecraft can serve as an effective vehicle for teaching mathematics and computing in the field of STEM education, and can consistently stimulate students' learning motivation and engagement. When instructional design is purposefully constructed, it can improve students' four competencies – spatial reasoning, geometry, logic, and computational thinking – within measurable ranges. This interactive platform allows students to apply mathematical thinking in real-world scenarios since mathematical concepts are presented to them through interactive, hands-on tasks in an environment that is both familiar and attractive. The mixed results generated by a small number of independent studies in this field do not act as counterevidence to this core claim. Instead, they emphasize that improvements in these competencies can only be observed if the three elements of learning tasks, instructional scaffolds, and assessment are properly aligned. Future research should be more rigorous, span longer time periods, and feature clear study designs to unlock the full potential of Minecraft in STEM education.

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Contact addresses:

Ana Mirković Moguš
Faculty of Education
Josip Juraj Strossmayer University of Osijek
Cara Hadrijana 10, 31 000 Osijek, Croatia
e-mail: amirkovic@foozos.hr

Željko Gregorović
Faculty of Education
Josip Juraj Strossmayer University of Osijek
Cara Hadrijana 10, 31 000 Osijek, Croatia
e-mail: zgregorovic@foozos.hr

Minecraft kao matematičko igralište: Sustavni pregled obrazovnih primjena, računalnog razmišljanja i matematičkih spoznaja

Ana Mirković Moguš* i Željko Gregorović

Fakultet za odgojne i obrazovne znanosti, Sveučilište Josipa Jurja Strossmayera u Osijeku, Hrvatska

Sažetak. Učenje temeljeno na igrama (GBL) sve se više prepoznaje po svojoj sposobnosti da poboljša angažman učenika i konceptualno razumijevanje. Među digitalnim alatima posebno se ističe Minecraft kao istaknuta platforma koja potiče kreativnost, suradnju i rješavanje problema kod učenika, razvijajući pritom ključne kompetencije. U obrazovanju iz matematike, gdje se učenici često suočavaju s izazovima u razumijevanju apstraktnih pojmova, Minecraft nudi dinamično trodimenzionalno okruženje izgrađeno oko matematičkih sadržaja, omogućujući učenicima istraživanje prostornog razmišljanja, geometrije, mjerenja i obrazaca kroz igru, istovremeno angažirajući i one učenike koji u tradicionalnim nastavnim okruženjima pokazuju manjak motivacije. Ipak, istraživanja o učinkovitosti Minecrafta kao obrazovnog alata, njegovim kognitivnim učincima te usklađenosti sa standardima kurikuluma još su uvijek ograničena. Cilj ovoga rada je analizirati rastuće istraživačko područje u presjeku Minecrafta, matematike i informatike. Analiza je provedena slijedeći PRISMA smjernice i primjenom narativne sinteze s kvalitativnom analizom sadržaja kako bi se identificirale ključne teme, nalazi i izazovi prikazani u relevantnoj literaturi. Rezultati pokazuju da Minecraft može olakšati različite obrazovne ishode, uključujući razvoj kreativnosti, timskog rada, kritičkog mišljenja, motivacije, aktivnog učenja i zadržavanja znanja. Pregled također ukazuje na postojeće praznine u empirijskim istraživanjima i usklađenosti sa standardima kurikuluma. Dobiveni nalazi pridonose dubljem razumijevanju obrazovne vrijednosti Minecrafta te postavljaju temelje za buduća interdisciplinarna istraživanja u rastućem području učenja temeljenog na igri i računalnog obrazovanja.

Ključne riječi: učenje temeljeno na igrama (GBL), Minecraft education, poučavanje matematike, poučavanje informatike, angažman učenika

Exploring Digital Resources as an Assessment System: Who is the Feedback Intended For?

Katarina Širanović[1]*, Goran Trupčević[2] and Dubravka Glasnović Gracin[2]

[1]Primary school Tituša Brezovačkog, Zagreb, Croatia
[2]Faculty of Teacher Education, University of Zagreb, Croatia

Abstract. Automated feedback is a feature of digital tasks in digital textbooks, which not only supports learning but also helps teachers monitor students' progress. This paper examines which resources teachers find helpful for obtaining feedback and how different forms of feedback from digital resources impact mathematics teachers' ability to monitor students' progress. The study involved six middle-grade mathematics teachers, with whom semi-structured interviews were conducted. The findings reveal that teachers receive feedback about their students' progress from digital textbooks, digital tools, and virtual classrooms. Teachers use digital tools to obtain extended feedback that helps them monitor students' progress, as digital textbooks provide only basic feedback, which is insufficient for gaining deeper insight. In view of the above, there is a considerable potential for pedagogical resources to be designed and existing materials to be enriched to fulfill an educative function for teachers as well.

Keywords: automated feedback, assessment system, digital textbooks, digital tools, monitoring students' progress

1. Introduction

In recent years, digitalization has increasingly become an integral part of education (Bates, 2022; Singh & Thurman, 2019). In this context, numerous potentials of digitalization emerge, including the way mathematics is taught and learned. One of the key transformations of the educational process is a shift from printed to digital textbooks, which brings new educational dynamics (Pepin et al., 2015). In mathematics education, textbooks play a fundamental role (Gueudet et al., 2012). A mathematics textbook is considered a valuable teaching resource for teachers

*Corresponding author.

and students for teaching and learning inside and outside of the classroom (Pepin et al., 2013). Traditionally, printed textbooks have long been considered the major conveyor of tasks for learning mathematics (Glasnović Gracin & Domović, 2009; Pepin & Haggarty, 2001). In Croatia, research has shown that textbooks are mainly used as a source of tasks for students, for teachers' preparation, or for giving homework to students (Glasnović Gracin, 2011; Glasnović Gracin & Domović, 2009). However, traditional printed textbooks lack the flexibility to adapt to individual learners or provide immediate and detailed feedback on students' performance. These limitations can be overcome with digital textbooks as an evolving structured set of digital resources that "encompass the mathematical content of the traditional textbook, but they also include diverse supplementary tools, in particular those that support the learning of the mathematical content" (Pepin et al., 2015, p. 644). According to Pepin et al. (2015), there are three types of digital textbooks: a) integrative digital textbooks (digital version with hyperlinks), b) evolving digital textbooks (allow teachers to author and modify content), and c) interactive digital textbooks (contain tasks and interactives that can be linked and combined).

Digital textbooks are designed to enhance learning mathematics by offering many potential benefits such as interactivity, dynamics, personalization, feedback, and cooperation (Choppin et al., 2014; Glasnović Gracin & Krišto, 2022). Feedback, as one of the key features of digital textbooks, is defined as "information provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding" (Hattie & Timperley, 2007, p. 81). Ramaprasad defines feedback as "information about the gap between the actual level and the reference level of a system parameter which is used to alter the gap in some way" (Ramaprasad, 1983, p. 4). It is considered a key element in formative assessment and one of the most influential learning factors (Shute, 2008). Black and Wiliam (2009) claim that "practice in a classroom is formative to the extent that evidence about students' achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction" (p. 9). Shute (2008, p. 154) defines formative feedback as "information communicated to the learner that is intended to modify his or her thinking or behaviour for the purpose of improving learning." Shute (2008) categorizes various types of feedback according to their complexity like verification, correct response, try again, error flagging, elaborated, attribute isolation, topic contingent, response contingent, hints/clues/prompts, bugs/misconceptions, and informative tutoring. Digital textbooks have the potential to incorporate assessment, offering feedback to students and performance data to various stakeholders, such as teachers, parents, or administrators (Choppin et al., 2014).

Many studies have investigated the effectiveness of feedback for learners, examining aspects such as feedback complexity, timing, or feedback content (Hattie & Timperley, 2007; Rezat, 2021; Shute, 2008; Van der Kleij et al., 2015). Automated feedback has been shown to offer many benefits in mathematics education for learners. For instance, it positively impacts students' performance in mathematical reasoning tasks in geometry, and it supports learning overall (Jang et al., 2016; Kaplar et al., 2022). Moreover, there are benefits of receiving automated feedback from digital textbooks for teachers as well, like giving teachers insights into their students' learning processes and challenges, understanding students' progression in general, and adjusting their teaching objectives accordingly (Almahdi

& Swidan, 2024; Panero & Aldon, 2016). In such conditions, assessment becomes formative and can efficiently contribute to students' learning (Panero & Aldon, 2016). Research has shown that extended feedback from digital textbook, which includes complete statistics such as a list of correct and students' incorrect solutions, and items that students did not solve, is considered a valuable source for teachers that leads to a change in teachers' understanding of their students' knowledge (Almahdi & Swidan, 2024). Automated feedback, as one of the key potentials of digital textbooks, not only supports learners but also transfers part of the responsibility for providing feedback from the teacher to the digital resources (Rezat, 2021). This shift allows teachers to focus more on interpreting the summarized results of students' work and adapting their teaching strategies accordingly.

Although automated feedback is considered a digital potential of digital textbooks, its implementation in the Croatian education context leaves open questions. Previous research has shown that geometry tasks in digital textbooks do not fully utilize digital potentials, particularly in dynamics, interactions, and personalization (Glasnović Gracin & Krišto, 2022). In Croatia, automated feedback is present in digital textbooks as it is required in the curriculum (Ministry of Science and Education [MZO], 2019), but there is a lack of other potentials of digital textbooks, such as formative assessment (Glasnović Gracin et al., 2025). In digital textbooks, automated feedback most commonly appears in the form of verification in simple tasks, whereas exploratory tasks more often provide learners with hints (Glasnović Gracin et al., 2025). Even though automated feedback is designed to take the responsibility from the teachers for providing feedback (Rezat, 2021), in the Croatian context, it is not fully utilized in that way (Glasnović Gracin & Trupčević, 2025). There is a notable lack of studies examining the specific role of automated feedback in supporting teachers to track learners' progress. Therefore, an open question remains as to how automated feedback can be used as a formative assessment and help teachers to monitor students' progress.

This study aims to address the gap by investigating the impact of different forms of automated feedback from digital resources on mathematics teachers' ability to monitor students' progress.

Therefore, the following research questions arise: (1) In what ways do teachers receive feedback from digital resources that help them monitor students' progress? (2) How do different forms of feedback help mathematics teachers monitor students' progress?

2. Theoretical framework

Van der Kleij et al. (2015) classified various types of feedback based on the distinction made by Shute (2008), which includes knowledge of results (KR), knowledge of correct response (KCR), and elaborated feedback (EF). According to Van der Kleij et al. (2015), *knowledge of results* refers to whether the answer is correct or not, without providing additional information. *Knowledge of correct response* involves revising the student's incorrect response by providing the correct one, while *elaborated feedback* is an additional characteristic of feedback that can be combined with other types of feedback in the form of hints. The focus of this study

will be on a specific type of feedback, namely verification feedback as described by Shute (2008), which aligns with the concept of knowledge of results according to Van der Kleij et al. (2015). This form of feedback relies on a simple indication of whether an answer is correct or not, without providing additional guidance or explanations (Shute, 2008). Specifically, this study explores how teachers can benefit from simple verification feedback summarizing students' overall performance.

Choppin et al. (2014) created a typology for analyzing digital mathematics programs, which highlights three themes that reflect the interconnected relationships between resources, learning, and teaching: students' interactions with the program, curriculum uses and adaptations, and assessment. Student interaction with the program refers to the activities and resources students engage in within the program. Curriculum use and adaptations refer to how teachers utilize and modify the program's tools and resources to sequence, organize, and design lessons based on their instructional needs. Assessment refers to the system's ability to create, record, store, analyze, and report the results of online assessments and generate and transmit them to multiple audiences. Choppin et al. (2014) created a framework to analyze assessment systems, which identifies four core functionalities of assessment systems: creating assessments, recording and storing results, generating dashboards or other data summaries, and producing and distributing reports to multiple audiences. The basic descriptions of these functionalities are provided in Table 1. This framework forms the foundation for the study presented in this paper. Since this paper focuses on automated feedback from digital tools for teachers, the possibility of generating and sending results to multiple audiences will not be discussed.

Table 1. Framework for analyzing assessment systems (Choppin et al., 2014).

Functionality	**Description of functionality**
create assessments	ability to select and store items
record and store results	programs' capacities to automatically record and store the results
generation of dashboards or other data summaries	transmitting the results to the teacher
generate and send reports/results to multiple audiences	ability to transmit the results to external audiences

According to Brown and Edelson (2003, p. 1), "teachers create, adapt and improvise with instructional resources most of the time", and because of this, they identify teachers as designers. Accordingly, they distinguish three types of utilization of teachers' resources: offloading (where teachers significantly rely on curriculum materials like textbooks), adapting (where teachers use and modify parts of curriculum materials by adding their own design elements), and improvising (where teachers do not depend on textbook content).

3. Method

For the purposes of this study, a qualitative research approach and a descriptive method were employed. As part of a broader investigation into the characteristics

and use of digital textbooks in Croatia, a multiple-case study approach was applied. The study involved six middle-grade mathematics teachers from the Zagreb area who use digital mathematics textbooks. The teachers have volunteered to participate after 78 schools had been informed about the study. The research data consisted of semi-structured interviews, which were conducted to explore how teachers perceive feedback from digital materials for tracking students' progress. For the purposes of this study, knowledge of (students') results was examined through two forms: simple and extended. The simple form refers to providing a summative overview of the results after a student completes tasks, while the extended form encompasses the simple form but also includes an analysis indicating which tasks the individual solved correctly and which were incorrect. Key questions addressed in the interview were: *Do you use digital resources for providing or receiving feedback? Which ones and how? In what ways do you use automated feedback? How do you perceive automated feedback from digital textbooks and digital tools?* Transcribed interviews were analyzed using qualitative content analysis with an inductive approach according to Mayring (2000). Based on the available data from interviews, categories were inductively derived, after which they were revised and condensed into main categories, and finally, they were examined in the context of reliability.

4. Results

4.1. Feedback from digital resources for teachers

From the analysis of the collected data, three key categories of digital resources emerged through which teachers receive feedback: digital textbooks, digital tools, and virtual classrooms. Some of these are perceived as sources of feedback just for students, while others are seen as sources of feedback both for students and teachers. However, most teachers use them primarily in a frontal way due to a lack of material resources (computer labs, tablets). As a result, digital textbooks and virtual classrooms cannot be used for assessing all students during lessons, whereas some digital tools do not require as much equipment and can therefore be used for evaluation.

Creating assessments. Teachers highlight the advantages of using tasks from digital textbooks, particularly emphasizing the opportunity for students to independently practice and verify the accuracy of their answers, thereby reducing the need for teachers to individually check their results. While digital textbooks do not allow mathematics teachers to create their own assessments, teachers can select pre-existing tasks that students are required to solve during the lesson. This allows some degree of customization, but it still limits teachers' control over assessment design. The interviewed teachers most frequently mentioned tools such as Edulastic, Google Forms, Plickers, and Quizizz for creating their own assessments. These tools offer greater flexibility, allowing teachers to design assessments.

T1: I used Edulastic before, which I think is brilliant. (. . .) It's a great platform because you can really create very neat tests and analyses, and it provides real-time feedback, which is super helpful for me.

T2: I use Plickers most often because it's convenient and relatively quick: I record it, they see which answers are correct, and I can see them too.

T4: So we created a series of assessments using Google Forms, which the teacher can copy and share with their classes, and then both the student and the teacher receive feedback on how the class performed.

4.2. The role of feedback in monitoring students' progress

To answer the second research question we look at how teachers use and view recording and storing results and generating dashboards or other data summaries by the three categories of digital resources.

Recording and storing results. Teachers point out that digital textbooks typically provide only a summary of results.

T1: As I said, the most obvious thing I see is whether you have worked or not. But I can achieve that without a digital textbook as well.

T5: So, it is possible, especially in those sections where, at the end, there is a test to check how accurately the student has solved something.

Teachers mentioned digital tools as resources they use to obtain extended feedback on students' achievement. In contrast to the simple feedback obtained from digital textbooks – which only provides a summative overview of the number of correctly performed tasks relative to the total number – digital tools offer the potential for more comprehensive feedback.

T4: Some tools I presented during those workshops, like Plickers, which I mentioned earlier, as well as other tools we introduced to teachers, provide feedback both to the student and to the teacher. For instance, Google Forms can be organized similarly. (. . .) For example, one such tool that functions in this way is Quizizz.

Teachers also mentioned the possibility of tracking students' progress by connecting them to virtual classrooms. Teachers are aware that by doing so, they could receive more detailed feedback on each student, but they do not actually use this in practice.

T6: There is the option to connect them all, and for me to somehow link them, but as I said, I haven't done that part yet.

Teachers who use virtual classrooms are not satisfied with the quality of the feedback received and the ability to track students' progress in more detail.

T1: It's really in that part of the interpretation because, honestly, I expected more from the virtual classroom. As soon as I read that it's possible to create it, I thought there would be more concrete tracking of the students, so that we could really read more from their progress. Because that's actually the most important thing for me – to see what they're doing with ease, what we can skip, and what we should pay attention to.

Generation of dashboards or other data summaries. Participants noted that the feedback they receive from digital tools includes detailed statistical analyses for

each student, showing which specific tasks the student completed and which they did not, the time spent on each question, and a summative breakdown of the tasks each student managed to solve, while digital textbooks provide only a summative overview of correctly completed tasks without any in-depth analysis which would help teachers to monitor students' progress.

The participating teachers' statements regarding the use of automated feedback from digital resources were gathered under two themes: a simple form of knowledge of results from digital textbooks and an extended form of knowledge of results from digital tools.

Teachers argue that the simple feedback provided by digital textbooks is insufficient for obtaining more detailed insight into each student's knowledge. Teacher 1 expressed dissatisfaction with the simplicity of feedback from digital textbooks, noting that they provided little more than basic evidence of student engagement, rather than a deeper understanding of their knowledge.

T1: They have those XP points, or whatever they call them, that students collect, but they don't really tell me exactly how much they actually knew or didn't know. It's more of a competitive element for them, like a game – who collected more points, who completed more tasks. But when it comes to feedback, I'd like it to have more meaning so I can clearly see what they did and didn't do.

The lack of extended feedback in digital textbooks led teachers to use supplementary tools that provide more detailed insights into individual students' knowledge. During the interview, three teachers expressed that they use digital tools in their mathematics instruction, which provide extended feedback. Specifically, they use a tool where they receive feedback on which tasks each student answered correctly and where they made mistakes, and they stated that this type of feedback gives them a broader picture of each student's performance. They mentioned using tools such as Plickers, Edulastic, Quizizz, and Google Forms. One teacher stated that she uses digital tools because, in addition to receiving extended feedback on each student, she obtains it within a very short time frame.

T1: I used Edulastic before, which I find absolutely brilliant, truly brilliant. (. . .) It's a great platform because you can create really nice tests and analyses since it provides real-time feedback, even for me. Students can work on it, and I can see everything – how many times they made mistakes, what they knew, what they didn't, and how much time they spent on each question. It provides comprehensive statistics, and since I love statistics, I find it extremely, extremely useful.

T4: We created a series of assessments using Google Forms, which the teacher can copy and share with their classes. This allows both the teacher and the student to receive feedback on how the class is performing.

T2: I mostly use Plickers because it's convenient and relatively quick. I scan the responses, and the students can see which answers are correct, while I can also see the results. It lights up green or red, so I immediately know who understands and who doesn't, as well as who answered correctly. It's quite handy.

The lack of the ability to create tasks and obtain detailed feedback from digital textbooks is compensated by teachers using digital tools in mathematics lessons. By using these tools, teachers have the opportunity to independently select tasks,

thereby receiving detailed feedback on each student. This approach can be interpreted as an example of teachers modifying parts of curriculum materials by adding their own design elements (*adapting*).

5. Discussion and conclusion

The study presented in this paper investigated ways in which teachers receive feedback from digital resources that help them monitor student progress, and the impact of different forms of feedback from digital resources on mathematics teachers' ability to monitor students' progress. It was found that teachers receive feedback about their students from digital textbooks, digital tools, and through virtual classrooms. Teachers use digital tools to obtain extended feedback that helps them monitor students' progress because, from digital textbooks, they can only receive simple feedback, which is insufficient for gaining deeper insight. Teachers are aware of the possibility of obtaining feedback for monitoring students' progress through virtual classrooms; however, they are either not sufficiently familiar with this feature, or it does not meet their needs in the context of receiving feedback.

Although automated feedback in digital textbooks is recognized as a digital potential that can reduce teachers' responsibility for providing feedback to students (Rezat, 2021), the results of this study suggest that teachers themselves are not entirely satisfied with the feedback they receive from such resources, as it remains too simple, which refers to providing a summative overview of the results after the student completes a task. Instead, they prefer more comprehensive feedback that further assists them in monitoring students' progress and provides more detailed insights than they usually receive through digital tools. Previous studies have shown that automated feedback from Croatian digital textbooks for mathematics is present in them as it is required in the curriculum (MZO, 2019), but there is a lack of other potentials such as formative assessment (Glasnović Gracin et al., 2025) which is why teachers often use digital tools to provide students with feedback, while also obtaining detailed feedback about each student's performance. As Ruthven (2008, as cited in Pepin, 2012) mentions, there is a large potential for pedagogical resources, such as digital textbooks, to be designed and existing materials to be enriched to fulfill an educative function for teachers. According to previous studies, detailed feedback from digital tools is considered to be the ultimate advantage for formative assessment (Robertson et al., 2019). Those features that teachers require can be found in digital tools because they offer various statistical measures that show learners' success, along with various data like the number of attempts or response time, which helps to monitor learners (Çekiç & Bakla, 2021). As is shown in previous research, extended feedback from digital textbooks, such as correct and students' incorrect solutions and items that students did not solve, features typically present in digital tools, is considered a valuable source for teachers that leads to a change in teachers' understanding of their students' knowledge (Almahdi & Swidan, 2024).

While digital textbooks already provide basic functionalities for obtaining feedback about student progress, there is considerable room for further improvement through the integration of more advanced formative features. To address some

of these issues, virtual classrooms have been implemented, where each student logs in with their personal data and completes tasks from digital textbooks. This allows teachers to access more detailed statistics; however, their full potential has not been realized yet.

While digital technologies offer increased possibilities for providing feedback, their effective use requires that schools be equipped with computers or tablets for students' individual work during lessons. When digital textbooks were first introduced, each student received a tablet, but today many of these devices are outdated or broken. Even when they were new, logistical issues such as charging, updates, and administration were not fully resolved. Additionally, a change in the Minister of Education one or two years after the reform reduced the emphasis on introducing new technologies. This shows that the development of digital textbooks and other digital resources must be accompanied by sustainable logistical support to ensure effective implementation in the classroom. A similar situation exists with virtual classrooms, which also require students to work individually on a computer or tablet, making logistical limitations one of the reasons for their limited use.

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Contact addresses:

Katarina Širanović
Primary School Tituša Brezovačkog
Španska 1, 10 090 Zagreb, Croatia
e-mail: siranovic12@gmail.com

Goran Trupčević
University of Zagreb, Faculty of Teacher Education
Savska cesta 77, 10 000 Zagreb, Croatia
e-mail: goran.trupcevic@ufzg.hr

Dubravka Glasnović Gracin
University of Zagreb, Faculty of Teacher Education
Savska cesta 77, 10 000 Zagreb, Croatia
e-mail: dubravka.glasnovic@ufzg.hr

Istraživanje digitalnih resursa kao sustava vrednovanja: Kome je namijenjena povratna informacija?

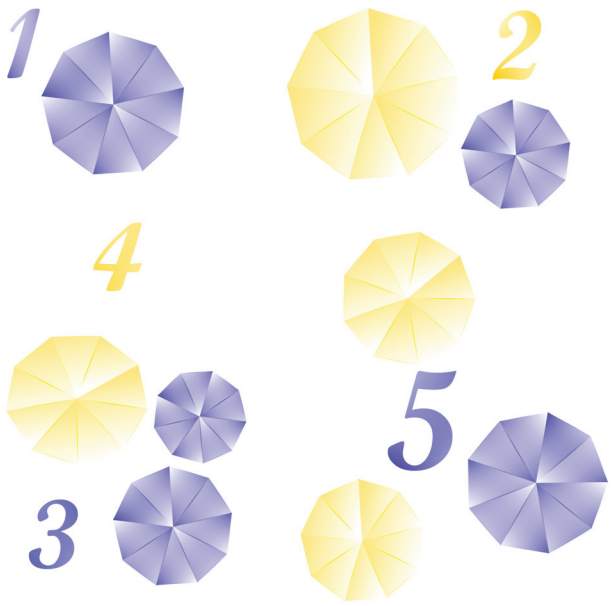
Katarina Širanović[1], Goran Trupčević[2] i Dubravka Glasnović Gracin[2]

[1]Osnovna škola Tituša Brezovačkog, Zagreb, Hrvatska
[2]Učiteljski fakultet, Sveučilište u Zagrebu, Hrvatska

Sažetak. Automatizirana povratna informacija vrijedan je potencijal digitalnih zadataka u digitalnim udžbenicima koji ne samo da podupire učenje, već i pomaže nastavnicima u praćenju napretka učenika. Ovaj rad ispituje koje resurse nastavnici smatraju korisnima za dobivanje povratne informacije te kako različiti oblici povratne informacije iz digitalnih resursa utječu na sposobnost nastavnika matematike da prate napredak učenika. U istraživanju je sudjelovalo šest nastavnika matematike predmetne nastave, s kojima su provedeni polustrukturirani intervjui. Rezultati pokazuju da nastavnici dobivaju povratnu informaciju o napretku svojih učenika iz digitalnih udžbenika, digitalnih alata i virtualnih učionica. Nastavnici se koriste digitalnim alatima kako bi dobili proširenu povratnu informaciju koja im pomaže u praćenju napretka učenika jer im digitalni udžbenici mogu pružiti samo jednostavnu povratnu informaciju, što je nedostatno za stjecanje dubljeg uvida. S obzirom na navedeno, postoji znatan potencijal za osmišljavanje pedagoških resursa i obogaćivanje postojećih materijala kako bi ispunili odgojno-obrazovnu funkciju i za nastavnike.

Ključne riječi: automatizirana povratna informacija, sustav vrednovanja, digitalni udžbenici, digitalni alati, praćenje napretka učenika

4.
Research, Teacher Development, and Interdisciplinary Perspectives



How the Mathematics and Children Research Group Has Contributed to the Field of Teaching and Learning Mathematics Over 20 Years – The 10th Anniversary of the Mathematics and Children International Scientific Colloquium in Osijek

Tatjana Hodnik

Faculty of Education, University of Ljubljana, Slovenia

Abstract. The *Mathematics and Children* research group has contributed to the field of mathematics teaching and learning over the last 20 years on various topics relevant to both the countries of the participants and the international research community. Nine scientific monographs were published in the framework of the International Scientific Colloquium. We will look at all the contributions in the published monographs and analyze which topics were addressed in the contributions, which were most typical in each period, and how they relate to the theoretical paradigms that prevailed in terms of the theoretical framework of the research in a particular period. In each monograph, we will cite or summarize specific parts of the texts that we believe are still relevant today or that appeal to the research community as much as they did in the past. The aim of this paper is also to provide a synthesis of the publications in each monograph, but also to bring our own view of the contributions from the perspective of the various educational policies that have guided mathematics education, both narrowly and more broadly. We believe that the 10th anniversary of the *Mathematics and Children* International Scientific Colloquium is an ideal time to review the work that has been done and to plan for the future.

Keywords: mathematics teaching and learning, theoretical paradigms, research community, changes through time in research, educational policies

1. Introduction

The aim of this paper is to highlight some contributions of the International Scientific Colloquium *Mathematics and Children* over time. Nine scientific monographs have been produced to date. I went through each monograph and tried to select the writings, the highlights, which, according to my knowledge, I can place in the present time or which I can comment on in the light of the knowledge I have developed in my career path. I am aware that this is one view and that someone else might think differently. I am grateful for this experience and hope that it will encourage you to reflect on the different paths we have taken over the 20 years of the International Colloquium *Mathematics and Children*. The contributions in each monograph are very diverse: we can read about education policies, practical examples of mathematics teaching, and insights into teaching and learning mathematics in its broadest sense. In reviewing the monographs, I realised that the Croatian research community is quite sympathetic to the study of geometry, textbook materials and the role of technology.

The monographs are distinguished by the fact that each contribution has a summary in the researchers' mother tongue, which contributes to the development of terminology in the researchers' native languages.

The monographs are also home to those who are no longer with us. Prof. Mirko Polonijo has written two articles in memory of two great Croatian researchers in the field of didactics of mathematics who have passed away: in the first monograph, he dedicated a text to Prof. Boris Pavković, and in the fifth to the founder of the Colloquium, Prof. Margita Pavleković.

Let us now look at my selected highlights of each of the monographs. I have the first one in print format (How to teach and learn mathematics, 2007; ed. M. Pavleković), and I have viewed all of the others on the conference website (10th International Scientific Colloquium *Mathematics and Children* (9–10 May 2025): `https://indico.mathos.hr/event/14/page/25-previous-monographs`).

Here is a list of all nine monographs of the International Colloquiums *Mathematics and Children* from 2007 to 2024:

2007 How to teach and learn mathematics (Ed. Margita Pavleković)

2009 Learning outcomes (Ed. Margita Pavleković)

2011 The math teacher (Ed. Margita Pavleković)

2013 Higher goals in mathematics education (Eds. Margita Pavleković, Zdenka Kolar-Begović, Ružica Kolar-Šuper)

2015 Mathematics teaching for the future (Eds. Zdenka Kolar-Begović, Ružica Kolar-Šuper, Ivana Đurđević Babić)

2017 Mathematics education as a science and a profession (Eds. Zdenka Kolar-Begović, Ružica Kolar-Šuper, Ljerka Jukić Matić)

2019 Towards new perspectives on mathematics education (Eds. Zdenka Kolar-Begović, Ružica Kolar-Šuper, Ljerka Jukić Matić)

2022 Advances in research on teaching mathematics (Eds. Zdenka Kolar-Begović, Ružica Kolar-Šuper, Ana Katalenić)

2024 Effective teaching and learning of mathematics through bridging theory and practice (Eds. Zdenka Kolar-Begović, Ružica Kolar-Šuper, Ana Katalenić)

2. Highlights from monographs through the years[1]

2007 How to teach and learn mathematics

The first edition does, of course, require a little more attention. In Croatia, more precisely in Osijek, the international publication space for mathematics education had just opened.

The topics addressed in this monograph are mostly still relevant today: differences between students (gifted and those with dyscalculia), problem solving, technology in education (smart boards, dynamic geometry), curriculum issues, teacher competences, artificial intelligence (an expert system for diagnosis of giftedness), basic knowledge of mathematics, mathematical language, the relationship between mathematics and mathematics education as scientific disciplines, and new topics of the time in primary schools (statistics, combinatorics and probability), which is related to the PISA introduction of mathematical literacy at the end of the twentieth century. We can also read about some specific examples of teaching geometry with materials such as origami and LEGO blocks.

Each of us had the opportunity to publish a topic that we were working on or had come across at the time.

I would like to highlight Kurnik's contribution *The Scientific Frameworks of Teaching Mathematics*, in which it is possible to recognise the wonderful interplay between mathematics and mathematics teaching. The author reminds us that the basic thought processes in science are analysis, analogy, abstraction and concretisation, generalisation and specialisation, and induction and deduction, which he illustrates vividly at different levels of education. These processes must not be neglected if we are to maintain mathematics learning at a level that can reasonably be expected. Without them, mathematics cannot develop students' conceptual thinking, which is crucial for the acquisition of mathematical knowledge and the development of conceptual thinking in general. This is also important in other areas, which is why mathematics is found in most areas of human life and creativity.

[1] Quotations or other parts of the text within individual monograph descriptions are taken from the monograph that is the subject of the description and can be seen from the section title, so citation is only by the surname of the first author. The section headings are the titles of the monographs from the respective years. Some quotes from the editorials of the monographs do not include page numbers because the editorial pages in the monographs do not have page numbers.

It is surprising to read about artificial intelligence (AI) in the very first monograph. Pavleković et al. evaluated an artificial intelligence expert system for classifying children as potentially gifted. The results showed that an AI system that includes more components of giftedness in its base classifies more children as potentially gifted. The authors concluded by noting that research can focus on possible improvements to the software tool by incorporating other AI techniques such as neural networks, general algorithms and intelligent agents.

I will also highlight the value of knowledge in the age of technology, which is equally or even more important nowadays. Rukavina (p. 140) states:

"It seems as though the current conflict between supporters of mathematics and those who are trying to reduce its content in schools is additionally exacerbated by the technological development. 'The supporters' believe that the technological development has intensified the need for mathematically skilled personnel, while the latter claim how do we not even have to know the multiplication table anymore because of the existence of various calculating devices. This conflict is a result of identifying minimal and basic knowledge of mathematics, i.e. minimal and basic mathematical literacy."

There are many different keywords in this monograph, covering practically everything that one could think of in the field of teaching and learning, but teaching mathematics is the one that comes up most often. Other keywords are mathematics, teaching with understanding, problem solving, motivation, gifted children, difficulties in learning mathematics, national curriculum, mathematics classroom, smart board, dyscalculia, assessment, evaluation, geometry, competence of prospective teachers, popularisation of mathematics, partnership among faculties, schools and families, basic knowledge of mathematics, teacher training, active participation, etc.

Numerous issues had accumulated over time, and we sought to bring them to the forefront of the discussion.

2009 Learning outcomes

In the introduction to the monograph, we read that:

"Learning outcomes are statements that describe what pupils/students needs to know, understand, and be able to do after they have successfully finished an educational cycle. Learning outcomes are coherent with measurable level descriptors in national and European frameworks." (Pavleković, p. 3)

Learning outcomes are mainly discussed in relation to changes in study programmes that were aligned to the Sorbonne Declaration 1998, the Bologna Declaration 1999, etc. Nowadays, due to the so-called Bologna process in higher education, study programmes have learning outcomes rather than objectives. These learning outcomes are further subdivided into knowledge and understanding, application, reflection and transferable skills. Knowledge, which includes content knowledge, skills and values, is the end result of a process defined by learning outcomes. We

consider that learning outcomes define the results of the teaching process more broadly and are a major contribution, at least for the processes of knowledge acquisition in study programmes.

The Programme for International Student Assessment (PISA) began in the year 2000. It was developed by the Organisation for Economic Co-operation and Development (OECD) to evaluate education systems worldwide by testing the skills and knowledge of 15-year-old students in reading, mathematics and science. PISA has had a specific impact on the understanding of mathematical literacy and which areas of mathematics should be developed in schools. In this monograph, Glasnović Gracin (p. 58) discusses tasks from PISA 2006 regarding:

"... only one-fifth of given PISA test items are common in Croatian mathematics education;

... more than one-third of given test items in PISA assessment in 2006, a student needed to apply the knowledge of statistics and probability...;

... statistics and probability were introduced to Croatian curriculum for the first time in fall 2006 and to a very limited extent;

... PISA program suggests a new approach to functions that could be embedded in our mathematics classes, but also in the physics education, and any other subject where students use the interpretation of graphical representations."

The latest Slovenian curriculum reform (2025) for mathematics reflects the latter: a greater emphasis on understanding the concept of function and integrating mathematics with other subjects. This certainly makes sense to some extent, but the question that arises here and that we need to address is what we lose by emphasising functions in different contexts. In our case, we have given up detailed treatment of the linear function. At this point, we can ask ourselves about the direction in which we are building school mathematics by introducing new "improvements" or adaptations with each reform. It is evident from our latest reform that we are mainly "afraid" of making mathematics too demanding for students.

Related to PISA, this monograph includes articles dealing with the topics of statistics, combinatorics and probability. These topics were most closely associated with mathematical literacy in the early PISA context. Mathematical literacy has most often been interpreted as a competence that empowers individuals not to be an easy target for various media.

If we look at the mathematical literacy tasks (Figure 1) from the last PISA assessment (2022), we can ask ourselves what needs to be developed in mathematics lessons in order for students to be successful. We also need to enquire how these things should be developed and what else we need to do in order to do well as a country. There are several different perspectives on this internationally, ranging from ignoring the PISA results to completely changing the curriculum. More than the results themselves, I believe that the key is to analyse students' problem-solving and their misconceptions, and to consider the actions we need to take in order to help students to overcome difficulties and become better problem solvers in general.

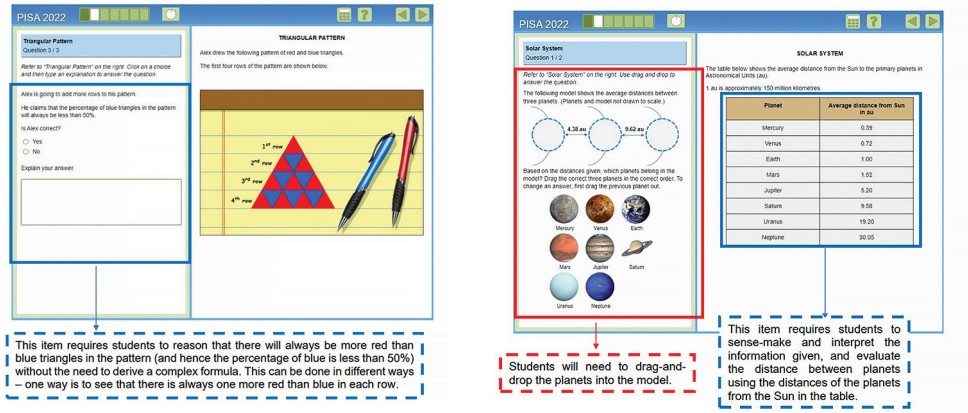


Figure 1. Some examples of mathematical tasks from PISA 2022 (OECD (2023), PISA 2022 results)

This monograph is also characterised by new keywords in addition to those included in the 2007 monograph: in addition to learning outcomes, the keywords include labour market, action research, interdisciplinary connections, "hands-on" activity, new methods in mathematics teaching, ICT, mathematical competences, metacognition, self-regulation and taxonomy.

Regarding taxonomies, we can only point out that at the time of the publication of the second monograph in 2009, Bloom's taxonomy had already been revised (first published in 1956, revised in 2001) and the Structure of the Observed Learning Outcomes (SOLO taxonomy), developed by Biggs and Collis in 1982, was in the process of being established. The latter has not really become established in our educational space (i.e., in the countries of the conference participants), but it is mentioned in one of the papers.

So, at the turn of the millennium, decisive changes were taking place: PISA, learning outcomes, competences, new taxonomies.

We cannot, however, avoid the fact that at that time, changes in the teaching of mathematics and new guidelines for teachers on how to teach were being announced. In the article by Hodnik (p. 24), we read a summary of Sylwester (1995) giving guidance to mathematics teachers:

"Organize cooperative learning (emphasize the social aspect of the experience). The emotional element of learning is important. Teach what is important to know or understand. Employ technology. Work in a team. When treating problems take into consideration all intelligences."

This goes hand in hand with the fact that teachers have been given instructions on how to teach over and over again, from the very beginning of the development of didactics in mathematics. Sometimes these instructions are contradictory and can lead to slogans, such as "active learning is important", "the learner must be at the centre", "students should learn in groups", "the teacher is the director, the organiser of the lesson", "the learner constructs knowledge himself", etc. These

short formulas have the power to persuade, to encourage people to act against or for something, and to legitimise or discredit a particular educational practice or theory (Kodelja, 2023). The more effective and persuasive the slogans are in this respect, the greater the risk that they will increasingly replace thinking (Kodelja, 2023).

Terms such as explanation, consolidation, abstract thinking, individual work, effort, productive struggle, rigour, etc. are negatively valued and are therefore removed from discourse and use.

Important topics that were further addressed in this monograph can be summarised in two groups: 1) care for students who do not speak the language of instruction and require special education, as well as projects that will lead to improvements in the treatment of this problem and concrete solutions; 2) the decline of interest in the teaching profession, especially in science and mathematics.

Today, as in other countries, we in Slovenia have not yet found an adequate solution to the inclusion of non-native speakers in the classroom. The fact is that a significant number of these students have to attend schools with lower educational standards, which is problematic from two points of view: 1) a language barrier is not the same as an ability barrier, and 2) students who should be in this programme because of their lower ability are not able to attend due to the occupancy rate. In addition, the decline of interest in the teaching profession has never been as low as it is today. In the field of education, science and mathematics, for example, we read the following in a paper by Pavleković et al. (p. 33):

"The gained data in the year 2008/2009 are a clear signpost that a significantly bigger number of pupils choose social science studies rather than the natural science studies, in spite of the fact that the labor market requires education experts in the field of mathematics and informatics. The research results indicate the need for forming teams of experts who shall motivate pupils and students to direct their interests towards the activities needed on the labor market. The results will be helpful in the process of coordination and renewal of the curriculum of teacher education studies."

What has changed in these directions? How can we encourage young people to decide to study education, especially mathematics education, in which we observe falling interest in young generations globally? This is a major problem in society that needs to be addressed very carefully, with the inclusion of all decision-makers in the field of education and politics.

2011 The math teacher

In the preface, we read that PISA 2009 results showed that 27% of Croatian participants never read for pleasure and more than 57% say they only read to find information.

Pavleković's editorial goes on to point out that faculties for teacher education need to lead the way to modern student education without delay and with flexible

programmes and resources: mastering tutoring systems that can be used for learning support and the acquisition of new knowledge and skills, integrating multimedia into the classroom, and personalising the approach to the student.

A question is raised in the editorial that is still very relevant today:

"Are we sure that the computer games currently on the market have been scientifically and professionally tested for possible use in (mathematics) teaching? Do the games available for entertainment really have a bad effect on children's health and their inner world? These and similar questions are being asked by teachers and parents who need scientific answers."

I believe that we have never been more concerned with these questions, and there is an increasing number of books that draw attention to screen addiction and the problem of distancing, i.e., the withdrawal of children from relationships and from themselves.

Another very important development in the Croatian context highlighted by Pavleković in her editorial is the progress in the development of a PhD programme in mathematics education, which would enable teachers to advance in their knowledge and positions.

"Likewise, faculties of teacher education are expected to create doctoral study programmes directed towards the scientific promotion of teachers. The academic community is required to give every teacher the opportunity for scientific advancement. This task became more realistic in 2009 when the National Science Council expanded its Statute on areas, fields and branches of science and arts. This statute includes in 1. Areas of natural science the field of 1.07. Interdisciplinary natural science accompanied by the branch 1.07.01 Teaching methodology in natural science."

I remember very well the efforts of Prof. Margita Pavleković to establish a PhD in mathematics education, both for teachers and for those interested in undertaking research in the field and thereby influencing education policies in this area. In 2009, the Ljubljana Faculty of Education initiated a doctoral programme that is open to students interested in the field of didactics of mathematics. Although the introduction of this programme was late compared to other European countries, it represents a major achievement in a region where mathematicians' discourse on mathematics education is quite specific. The first programme of this kind in Croatia has been running at the Faculty of Science, University of Split since 2020.

In this monograph, a significant number of contributions deal with teachers' knowledge, which is far from what we would expect. It is clear that teachers are agents of change if they are well educated, both in mathematics and in pedagogy. Another problem that is highlighted, as indicated in the previous monograph, is the perceived decline in interest in teaching.

Rukavina et al. (p. 14) point out in their paper that:

"Among subjects who do not consider the mathematics teacher profession as their potential future profession, 54.49% of them select the low salary as a factor that influences their views, and male subjects more frequently select this factor as

one of the reasons for indifference towards the mathematics teacher profession. This result can be partially explained by the divergence in gender roles, that is, traditional belief that men have to take care of the family."

The teacher's professional role is often clouded by his or her dependence on textbook materials, which do not necessarily follow the curriculum or the recommendations of the profession. Glasnović Gracin (p. 240) points out a similar problem:

". . . the Educational plan and program clearly emphasizes appliance of mathematics in everyday life. However, the textbook analysis shows the significant lack of context items. Therefore one can conclude that there is a discrepancy between curricular and textbook requirements in the intercultural domain."

Textbooks are the subject of research from many different angles: some would abolish them, there are ongoing reflections on how to improve them, e-materials are becoming increasingly popular, and there are questions about how textbooks shape the teaching process. Our students are already copying or looking at ready-made transparencies instead of the teacher creations on the blackboard, as teachers are well equipped with a variety of materials by publishers. Apple (1992) writes critically about this, saying that the best example of control is to be found in the extremely rapid growth of pre-prepared material for teachers: schools buy a complete set of materials for each subject (listing objectives by week, elaborate lesson preparations, activity suggestions, selected videos and websites, didactic approach, didactic material, a test-taking tool with which the teacher selects an objective and the software offers a set of tasks to test that objective, etc.). Apple has no doubt that this is an illustration of the devaluation of professional competence, which, paradoxically, is enthusiastically welcomed by teachers.

New keywords in this monograph in relation to the previous monographs are: active learning, creativity, technology, exploratory learning, classroom projects, mathematical experiments, constructivist approach to teaching and learning through information processing, and interactive constructivist approach to teaching and learning.

Although the role of technology is quite present, there are certain reservations about its use in the classroom.

Divjak et al. (p. 98):

"But we must not forget that pedagogy precedes technology and that setting educational goals comes before looking for appropriate ICT tools or other media. Setting technology in the first place might have an opposite effect and lead to saturation and loss of students' engagement. Therefore we have to be aware of the negative effects of using technology."

With regard to the use of technology, I think it is important to highlight the criteria for the decision to use it. Munđar et al. (p. 122) used the Analytic Hierarchy Process (AHP) technique for multi criteria decision making.

"The procedure for using the AHP can be summarized as: 1. Model the problem as a hierarchy containing the decision goal, the alternatives for reaching

it, and the criteria for evaluating the alternatives. 2. Establish priorities among the elements of the hierarchy by making a series of judgments based on pairwise comparisons of the elements. 3. Synthesize these judgments to yield a set of overall priorities for the hierarchy. 4. Check the consistency of the judgments. 5. Come to a final decision based on the results of this process."

It would also be useful to apply similar criteria when thinking about the role of AI.

In this monograph, the theme of problem solving is suggested, but different terms are used for this activity or approach, including experiment and discovery learning. Teachers are already being encouraged to take a different approach to teaching. Brueckler (p. 169) concludes the paper with the question:

"If you have the choice between a teaching method that requires 1) little effort, since you and everybody uses is since ages, although it has proven to be deficient and uninspiring both for the teacher and the student and 2) some effort, mostly because it is new and there are not yet many prepared teaching materials for this one, but promises better learning outcomes as well as more fun for both the teacher and the student, which method will you choose?"

In the conclusion of the monograph, we can read about the competences for modern teacher education, where the results of research show that the competences acquired by students in university programmes of study are somewhat lower than those of more practically oriented programmes of study.

Baranović et al. (p. 256) state:

"It is indicative that older teachers and teachers who graduated from teacher colleges tend to use contemporary teaching methods and teaching materials more often than younger teachers who hold a university degree (especially those who are at the age of 29 and younger). An exception to this is the use of ICT which is used significantly more often by younger teachers. These differences among teachers could be explained by a difference in length of teaching experience and different teacher education tracks: whereas older teachers have longer teaching experience and their teacher education gave them broader teaching knowledge and skills since they attended specialized teacher colleges, the younger teachers are less experienced and their teacher education took place at institutions (faculties) where the focus was subject mastery."

Although such conclusions are reasonable, they can lead to misinterpretations. We find that our students of two-subject programmes (mathematics and another subject) have considerably less practical training during their studies than prospective primary teachers. Despite the fact that students point this out again and again in their practice reports at our faculty, the programmes are unlikely to change in this regard. One reason for this is that they are university study programmes, which by definition are not practically oriented. In addition, there is a belief that increasing the proportion of practical teaching experience in schools would result in students acquiring insufficient subject knowledge. We are not yet at the point where we know how to execute and understand educational programmes with a meaningful

connection between theoretical knowledge and practical experience. This requires a major shift in the minds of all those involved.

2013 Higher goals in mathematics education

In the introduction, we read about contemporary themes that suggest that the key to learning mathematics is understanding, especially in the teaching of algorithms, which should not only be part of procedural knowledge, but also, or especially, conceptual knowledge. In addition to the mathematical knowledge of the teacher, pedagogical content knowledge is also emphasised, indicating that there is an emerging awareness of the importance of pedagogical competences in the field of mathematics teaching.

Thus, in the editors' introduction we read that:

"Mathematics education reforms call for decreasing the emphasis on complex paper-and-pencil computation drills and placing greater attention to mental computation, estimation skills, thinking strategies for mastering basic facts and conceptual understanding of arithmetic operations. Reform educators believe that such understanding is best pursued by allowing children to solve problems using their own understanding and methods. Under the guidance of the teacher, students eventually arrive at an understanding of standard methods. Such learning can occur in student-centered classrooms, using new, innovative, contemporary methods of teaching, and abandoning traditional transmission of knowledge. The challenges of contemporary teaching of mathematics can be overcome with continuing teachers' professional development."

Of course, we can also lose our way in this regard. According to Biesta (2022), there is a threat that in an age in which the written word is losing its priority to the image and every stimulus should be as short, dynamic, colourful and noisy as possible, classroom experiences seem outdated and boring, and that the price of this anachronism is huge. This leads to the conclusion that schools are becoming more irrelevant than ever to real life, and that teachers are losing their authority and their influence on children's lives. However, we have to be cautious when introducing innovations into schools, bearing in mind that with every innovation we also lose something; we have to judge whether the particular innovation is worth losing what works in schools. We need to be much more far-sighted, which is difficult in the face of the enthusiasm of those promoting change (sometimes simply for the sake of change itself).

This monograph raises one very important point of which we should be even more aware: the transfer of research results into practice is not easy, and it is important to realise that we (researchers and teachers) learn from each other, as the introductory thoughts make very clear:

"It takes time before they become known, so we researchers must become better at developing results in a form that makes them easier to use in practice, in the classroom. The aforementioned collaboration must be a two-way street:

practitioners should learn from scientists, but scientists should also learn from practitioners."

New keywords in addition to those already suggested in the previous monograph are: problem solving, conceptual and procedural knowledge, word problems, representations, algorithms, e-learning, distance learning, Moodle, and innovative approaches to teaching.

Another very important change from the previous monographs is the integration of examples of students' problem solving (pictures of their products) and records of dialogue with students when solving tasks or problems in the research paper. This contributes to a better understanding of how students learn mathematics.

The monograph is bursting with ideas about how to change teaching. Thus, we read very interesting contributions on how, for example, division of fractions could be taught in a more modern way, using contextual tasks instead of the traditional way. My personal opinion is that this approach is extremely interesting for those who know and like mathematics, but for the typical student, or for the student who struggles with mathematics, it is much more challenging. This is not to say that we should reject such approaches, but it is essential that we do not advocate them just because they are not traditional. We should not ignore the fact that memorisation and consolidation, which were understood or regarded as undesirable, are also important in mathematics. Fortunately, we now have sources that emphasise the importance of both direct and dialogical teaching (Hattie et al., 2017).

At the time of publication of this monograph, formative assessment of knowledge was becoming relevant. In her article on formative assessment, or rather on the method of communicating this assessment, Barišić (p. 148) writes that students do not approve of receiving feedback on their work via email.

"... students emphasized the negative aspects of communication through ICT pertaining to instructors. It would be desirable to examine the opinion of instructors on these issues to see whether these negative experiences grounded or not."

One student notes: "I support this type of communication (ICT), but in limited amounts, it is important to maintain face to face communication."

Horvat (p. 182) highlights the importance of socio-cultural aspects in teaching mathematics when she writes:

"Failures and accomplishments of teaching mathematics come intertwined with students' emotions and beliefs, which, interconnected with external factors such as family, society and school, affect student's success and further development. Considering social dimensions of mathematics education, successful teacher should develop adequate communicational, reflexive, social and emotional skills."

More recently, Ljungblad (2021) has defined this as pedagogical tactfulness, whereby the relationship is placed at the starting point on several levels: the micro level – the teacher-student relationship, questioning what it means to be a teacher and what it means to be a student as a starting point; the intermediate level – communication and organisation of the educational process at school by teachers and

school management; and the macro level – the social aspect or education system, as established by education policies, laws, research and global influences.

What does it mean to put relationships at the centre? It means that teachers can enable every student to speak with his/her own voice, in the present moment, with pedagogical tact in a face-to-face relationship. These questions and the resulting actions open up space for the subjectification of students.

A special place in this monograph is reserved for geometry. Geometry is a rather important field in Croatia, which, in my opinion, is mainly due to the awareness, as Alexander (2008) says, of the importance of teaching as an introduction to the canon: we impart knowledge that is distinguished in our civilisation. Of course, there are other goals of teaching geometry, which we will not mention here.

As noted in the introduction to this paper, a large proportion of research contributions in Croatia are devoted to the issue of teaching materials. In the last paper of this monograph, Tripković et al. conclude that teaching materials in Croatia are similar in structure to those in Germany. This is quite a natural consequence of understanding the nature of mathematics and didactics in Central Europe. Compared to, for example, English materials, which are much less structured, it is clear how we build knowledge in mathematics in our material, which is not necessarily something we should completely change.

2015 Mathematics teaching for the future

The editorial primarily highlights the international studies TIMSS and PISA. These two studies show the position of individual countries in terms of mathematical knowledge and mathematical literacy in the international arena. As stated above, mathematical literacy is a rather problematic concept, but for the purposes of international research, we follow the concept that is the consensus among experts in this field. Although we mentioned earlier that geometry is a particularly emphasised subject in Croatia, it turned out that Croatian students did not achieve the expected results in PISA 2009.

Glasnović Gracin (p. 21) summarises the lessons from PISA 2009 as follows:

"... connecting mathematical content with realistic and authentic contexts to a greater extent, greater use of mathematics in problem situations (not just to practise a routine, but to really solve a particular problem), more focus on reflective requirements and understanding mathematical concepts within textbook exercises, introducing a thorough approach to the idea of function as well as implementing it in authentic situations, having more focus on statistics and probability in compulsory education, especially through everyday interpretations.

These suggestions should be implemented not with the aim of getting better results in PISA assessment, but to truly improve mathematics education and to encourage students in reflecting on mathematical ideas."

Rather than particular ideas for teaching, there is a noticeable shift in this monograph towards building teaching models from which different approaches can be derived, enabling a broader understanding of teaching.

Only occasionally does the term constructivism appear in the monograph articles, and it is not among the keywords. One of the students who participated in the course Learning Mathematics as Construction Processes, in which students were required to reflect on their insights, wrote:

"In my opinion, almost everything can be learned by kids' own discoveries, assuming that one provides specific learning materials. Even concepts could be established by working on tutorial sheets. But I think it is good for kids if you bring some variety to teaching and don't withdraw yourself altogether. 'Let the kids just do their own thing' is not a healthy attitude in discovery learning." (Helmerich, p. 158)

As early as 2006, Kirschner, Sweller and Clark stated that constructivism "promotes" a teaching style with unguided or minimally guided instructions, and indicated that when students learn with minimal instruction, they become "lost and frustrated [...] and that minimally-guided approaches as practised through constructivist approaches ignore empirical studies that have shown that unguided instructions are not effective in learning environments". Here it is worth noting that the term "constructivist approaches" is very problematic: constructivism is an epistemological theory and its implications for teaching are in many cases very simplified.

When teaching is addressed in certain articles, it is divided dichotomously into traditional and contemporary. In a given situation, when we want to make a turnaround in teaching, a dichotomous approach may even make sense, but we must not forget that mathematics is about both the so-called traditional and the contemporary. Today, we are even more aware of this, as we have (hopefully) realised that by favouring so-called contemporary approaches to teaching, many teachers have lost their authority. Moreover, by directing teachers to focus on organising, directing and observing lessons and students' work, we have effectively removed them from their relationship with students.

The contributions in this monograph deal extensively with approaches to mathematics in the university environment. However, a significant difference from previous monographs is the chapter entitled *The influence of convictions, views, norms, emotions and attitudes of mathematics teachers on the efficiency of mathematics instruction*. Much of what happens in the classroom depends on the teacher's beliefs, including subjective theories about teaching and learning, which should not be overlooked. No matter how interesting the idea or how convincing the research results, it is always the teacher who decides how to act in a given situation.

Emotions and their study also play an important role in problem solving. For example, Kuzle in her study summarises the behaviour of students when solving problems. It is not just a matter of observing the strategies used by the student, but also examining how they feel about these strategies.

"The findings of the study showed that affective behaviours, such as perseverance, persistence, confidence, interest, and frustration occurred frequently during the problem-solving activity. These affective behaviours changed during the process of solving a problem and were related to participants' success when problem-solving." (Kuzle, p. 256)

Such findings coincide with the theoretical paradigm of the time; namely, the socio-cultural paradigm, which emphasises understanding the student as a human being in the broadest sense. Since the beginning of the new millennium, there has been increasing interest in addressing social factors in the teaching and learning of mathematics (Lerman, 2000). In contrast to epistemological, ontological and knowledge debates, which focused on the individual's acquisition of knowledge and explaining its relation to reality, the focus on social factors in (mathematics) education research has led to a so-called social turn. This concept refers to the emergence of theories that understand thinking, reasoning and the attribution of meaning as a consequence of social activity, emphasising the classroom climate and the socialisation process (Gutiérrez, 2013).

2017 Mathematics education as a science and a profession

Problem solving and technology play an important role in this monograph, and several studies examine the teaching and learning of mathematics in secondary school, in contrast to previous monographs, which focused on preschool and primary education. The entire monograph is mainly focused on issues in secondary schools and faculties. The use of technology in more complex topics, e.g., GeoGebra, Graph, computer algebra system, is highlighted. Students' attitudes towards mathematics also play an important role in this monograph. A new keyword that could be highlighted is tutoring in the university environment.

Babić et al. (p. 246) link tutoring and anxiety as follows:

"...it was revealed that students who possess a lower level of anxiety toward mathematics teachers are the ones who use mathematics tutoring services at university. For assessing and improving the quality of courses, it is useful for educators to know the profiles of students who use mathematics tutoring services and to be aware that students who do not have a negative attitude toward mathematics teachers use mathematics tutoring services at university."

In an article on mathematical anxiety, which aimed to develop a method for determining the level of mathematical anxiety in students, Đurđević Babić et al. (p. 255) summarise:

"Results derived from this research could serve to mathematics teachers as help in detecting students that have tendency to mathematical anxiety when observing some variables that are not evidently connected with it. Teachers could use these insights to raise the level of quality of their teaching by applying more individualised approach in their class and take into consideration different teaching and learning styles, which may help in creating more enjoyable and stimulating classroom environment."

This is true, but how can this be achieved with a large number of students in the lecture room and when students do not talk about their problems? While recognising that teachers can do a lot in this regard, we must also maintain sufficiently high expectations regarding student knowledge. Where is there room for change in higher education?

In the recent START project (Start – Supporting teachers who support student transition , `https://supportingstart.eu/`), which addresses the issue of student failure in the first year, we highlighted areas that need to be addressed in order for students to succeed. These areas are personal, social, academic engagement and university culture, with the key issue being the establishment of a culture of trust. When studying mathematics, students can face a double problem: fear of mathematics and fear of the maths teacher. This was evident in interviews with two of our students, which were conducted as part of the project. On the START website, there is a questionnaire that can guide first-year university teachers towards improving their work. How this will happen is, of course, another question. We do not mean to say that we do not strive to improve students' results in mathematics (sometimes even more than the students themselves); I simply want to point out that there is a silent assumption among a number of university mathematics teachers that students generally do not know much about mathematics and that there is not much that can be done about it.

Horvat (p. 264) offers a solution for overcoming mathematical anxiety in new teacher competency:

"In the context of the anti-anxiety curriculum, the need for a therapeutic dimension of math teachers becomes apparent in order to successfully direct negative student attitudes and beliefs about learning mathematics towards success, i.e. activities not restricted to negative aspects from the affective domain."

Since mathematics is an inherently abstract science, fear is also linked to the fact that students do not understand it. The solution does not necessarily lie in therapy, but in striving to achieve an understanding of abstract concepts, a good relationship between teacher and student, and an understanding that students learn at different rates. If we are patient enough and act appropriately, we can help students overcome their difficulties with mathematics and the associated anxiety.

2019 Towards new perspectives on mathematics education

This monograph presents entirely new content, i.e., the Anthropological Theory of Didactics (ATD), while other topics, primarily problem solving, the role of technology and teaching materials, continue from previous monographs. And, of course, geometry: some of the contributions on this topic are visually very appealing.

Regarding textbooks, tasks, conceptual problems and ATD, Milin Šipuš (pp. 176–177) summarises research in this area as follows:

"A mathematical body of knowledge is a product of human activity made to solve a problem or answer a question. The part that a researcher goes through

to find an appropriate solution includes studying the problem, researching other bodies of knowledge, transposing some knowledge to the given context and creating new knowledge to answer the question. The ATD postulate that students should acquire knowledge, new to their mathematics education, in the same manner as do the researcher and scientists. . . . The paradigmatic shift that ATD suggests is to change the mathematics curriculum from a list of objects to be covered to a set of generating questions that induce study and research path."

Contextual tasks have the potential to raise mathematical knowledge and open paths for new questions. I believe that the future is in a combination of a list of objects to be covered and contextual tasks.

A significant difference from previous monographs is the tendency towards an even deeper understanding of students' thinking.

Perhaps we could summarise here that the contribution to research of the role of technology complements the pedagogical triangle (Mirković Moguš, p. 338).

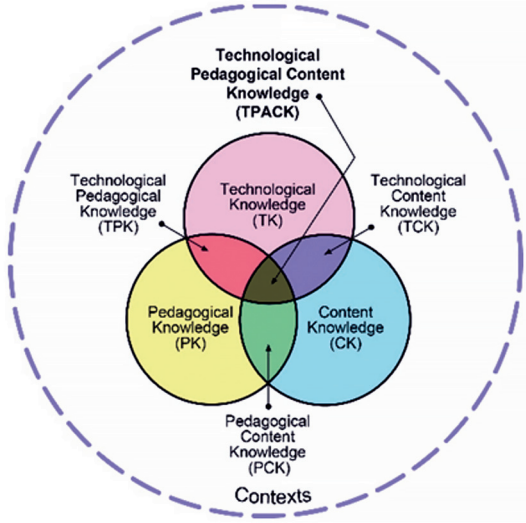


Figure 2. Technology as a complementary part of the pedagogical triangle (in Mirković Moguš, p. 338, reproduced by permission of the publisher, ©2012 by tpack.org.).

Since we have not yet said anything about the topic of measurement in mathematics (less relevant for some students, but quite challenging for others, especially when converting units) I would like to highlight a task that was given to first-year students to consider what applies to the lengths of paths (Kolar-Begović et al., p. 111):

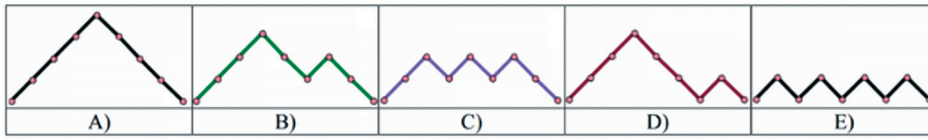


Figure 3. What can you conclude about the length of paths A, B, C, D and E? (Kolar-Begović et al., p. 111).

As mentioned several times above, research on the quality and role of teaching materials in Croatia is very topical and prevalent compared to other countries participating in the conference. It would certainly be interesting to know what impact this research has had on changing materials, or what teaching materials policy prevails in this area. Of course, we cannot ignore the role of capital, which is fighting for the textbook market. We have already mentioned the conclusions of the author Apple (1992) in this regard.

Let me conclude my highlights of the monograph with the issue of screen dependence. How appropriate is it for children to be exposed to screens? A study by Romstein (p. 360) revealed the opinions of preschool teachers and parents. Particularly interesting is the data on the appropriateness of screens after the age of three years. It seems that teachers are more aware of the consequences of screen addiction after the third year of life in comparison to preschool teachers: 28.8% of parents involved in the study stated that screens are not appropriate at this age, but more than half (52.9%) of preschool teachers expressed their disapproval.

Table 1. Disapproval of the use of screens (Romstein, p. 360).

Child's age	**Parents (%)**	**Preschool teachers (%)**
During 1st year of life	0.5	–
During 2nd year of life	16.5	–
During 3rd year of life	54.2	47.1
After 3rd year of life or later	28.8	52.9

In Slovenia (as well as elsewhere in the world – some countries are much more decisive in this regard), there are currently discussions on how to limit screen time for preschoolers, schoolchildren and young people in general, who are becoming increasingly dependent on screens. Many schools have already introduced a ban on mobile phones in schools. Another thread in terms of gaining knowledge is AI, which has begun to create the illusion that the individual no longer needs their own knowledge and skills, because we "have" "knowledge" that is immediately and constantly accessible outside ourselves.

2022 Advances in research on teaching mathematics

This monograph contains significantly fewer contributions than the previous one, which is an obvious consequence of the COVID-19 situation. It is certainly commendable that the conference was organised despite everything, and that the tradition was not interrupted.

The monograph could be described as a renewed effort to understand the teaching and learning of mathematics. We can conclude that the contributions focus on representations, misconceptions, problem solving and the use of technology. Peer interactions, which had not yet been addressed in previous contributions, could be

highlighted as a new topic. It is based on a socio-cultural paradigm that focuses on all forms of relationships in the teaching and learning of mathematics.

In terms of technology, we can observe a new dimension: connecting teachers and their practices in their home country and beyond (eTwinning).

The monograph concludes with two articles that discuss online learning during COVID-19. It is certainly true that no citizen project would have empowered citizens digitally as much as this situation did.

2024 Effective teaching and learning of mathematics through bridging theory and practice

In the most recent monograph, we could highlight an even narrower focus than in previous ones. The focus of the contributions is more specific, with more detailed explanations of what happens in classrooms, delving into various relationships. The themes are well framed in theoretical frameworks.

Within the theme of problem solving, which is a theme of all of the latest monographs, Fermi problems are included. Among them, one was particularly interesting: *How many people could participate in the demonstration at Ban Jelačić Square in Zagreb?* (Vranić, p. 109)

In the analysis of teaching materials, which is also present in this monograph, an analysis of a unique mathematical picture book is included, which confirms that it can be used to teach students about the different meanings of the number 0. This time, the contribution on teaching materials came from Slovenian authors, Manfreda Kolar and Mohorič, with a math picture book entitled 70 češenj (70 cherries).

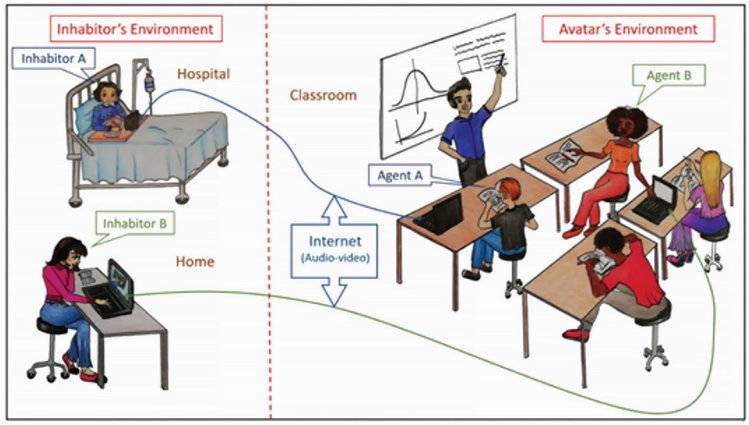


Figure 4. The hybrid model teaching (Duka et al., p. 383).

The class is illustrated as follows (Duka et al., p. 383): the teacher is in front of the blackboard, which is what some would call traditional teaching. I believe that it

needs to be emphasised again and again that the most important issue nowadays is not where the teacher is positioned in the classroom, but how *trustful and respectful teacher-student relationships* emerge over time.

Peer learning, which authors had discussed in the previous monograph, is upgraded in this issue, having taken on the new dimension of "avatar" learning. In our opinion, this is an interesting contribution to the current situation, as it introduces technology in a way that includes students who would otherwise be absent from class. The article discusses the Tablet-Human Hybrid Model of Avatar (Duka et al., p. 383), stating that: *"The model utilised a present classmate as an agent who paired his tablet with the tablet of an absent student via an audio-video connection. The agent's role was to fulfil the absent student's requests."*

3. Conclusion

The history of the International Scientific Colloquium *Mathematics and Children* is short but very rich in contributions, which can be read in the conference monographs. The contributions to the conferences have included topics relevant to the field of mathematics teaching and learning, reflecting education policies over time. I can say that, among other achievements, the colloquia have helped establish mathematics education as a discipline in the region. In conclusion, I would like to express my sincere gratitude to the organisers for their work and wish them continued success in organising conferences that provide a gathering place for the exchange of knowledge and critical views of various researchers in the field of mathematics education, and an opportunity to contribute to a better understanding of teaching and learning mathematics and, most importantly, of learners.

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Contact address:

Tatjana Hodnik
Faculty of Education, University of Ljubljana
Kardeljeva ploščad 16, 1000 Ljubljana, Slovenia
e-mail: tatjana.hodnik@pef.uni-lj.si

S katerimi temami je raziskovalna skupina "Mathematics and Children" v 20 letih prispevala k poučevanju in učenju matematike – 10. obletnica mednarodnega znanstvenega kolokvija "Mathematics and Children" v Osijeku

Tatjana Hodnik

Pedagoška fakulteta, Univerza v Ljubljani, Slovenia

Povzetek. Raziskovalna skupina "Mathematics and Children" je v zadnjih 20 letih prispevala k poučevanju in učenju matematike na različnih področjih, pomembnih za države udeležencev in mednarodno raziskovalno skupnost. V okviru konference je bilo izdanih devet znanstvenih monografij. Pregledali bomo vse prispevke v izdanih monografijah in analizirali, katere teme so prispevki naslavljali, katere so v posameznem obdobju najbolj aktualne in kako se povezujejo s teoretskimi paradigmami, ki so v določenem obdobju prevladovale v smislu teoretskega okvira raziskav. V posamezni monografiji bomo izposvaili določene dele besedila, za katere bomo presodili, da so aktulani še danes oz. da enako kot v preteklosti nagovarjajo raziskovalno skupnost, da se jim posvetijo. Namen prispevka je podati sintezo objav v posamezni monografij, hkrati pa vključiti lasten uvid v prispevke z vidika različnih edukacijskih politik, ki so usmerjajo matematično izobraževanje, tako v ožjem kot tudi širšem smislu. Verjamemo, da je 10. obletnica konference "Mathematics and Children" idealni čas za pregled opravljenega dela in načrt za nadaljnje.

Ključne besede: poučevanje in učenje matematike, teoretične paradigme, raziskovalna skupnost, spremembe v raziskovanju skozi čas, edukacijske politike

Insights into Self-Reflection and Self-Assessment of the Development of Teaching Competences in Mathematics Students

Vida Manfreda Kolar* and Adrijana Mastnak

Faculty of Education, University of Ljubljana, Slovenia

Abstract. Numerous researchers emphasize the increasingly complex and demanding role of teachers in today's rapidly evolving society, highlighting the importance of preparing future educators for lifelong learning. Consequently, teacher education aims to develop a broad range of competences, including subject-specific knowledge, pedagogical skills, and methodological approaches. Equally important is fostering awareness of personal beliefs about teaching, as these significantly influence how future teachers perceive and interpret learning situations and educational outcomes. To support this development, it is essential to provide future teachers with relevant content knowledge that can be effectively applied in their teaching practice, while also encouraging continuous improvement through self-reflection. This paper explores the role of student self-assessment and reflection on their own knowledge within the context of their evolving teaching identity.

The study sample comprises students enrolled in a two-subject mathematics degree program at the Faculty of Education, who, over the course of their studies, have acquired the necessary professional and didactic knowledge for teaching mathematics. The research focuses on an analysis of teaching practice diaries collected during two consecutive academic years (third and fourth year). Specifically, the study investigates two key areas: students' self-assessment of their competences for teaching mathematics, and their self-reflection on the mathematics lessons they conducted. Our primary interest lies in identifying areas where students perceive gaps or progress in their teaching competences and in understanding which aspects of their teaching they emphasize during reflection. Additionally, we examine potential differences in self-assessed competence between the third and fourth years of study. Through the analysis of teaching practice diaries, we aim to gain insight into how students perceive themselves, their roles, and their knowledge as future mathematics teachers. These findings will inform our ongoing efforts to support and empower students in becoming confident, competent, and reflective educators.

Keywords: mathematical competences, mathematics student, reflection on teaching, self-assessment, teaching practice

*Corresponding author.

1. Teachers' professional knowledge

The concept of teachers' professional knowledge has been the subject of numerous discussions in recent decades. The most influential is Shulman's model (1986), which emphasized the importance of content knowledge. According to Shulman (1986) understanding how teachers acquire, organize, and transmit content knowledge is central to effective teacher education. Teacher knowledge is not limited to mastery of subject matter but also encompasses pedagogical strategies and curriculum understanding. Shulman distinguishes between three key domains of teacher knowledge: subject matter content knowledge, pedagogical content knowledge (PCK), and curricular knowledge.

- *Subject matter content knowledge* is a teacher's deep understanding of the facts, concepts, and structures of a discipline, including both substantive structures (how concepts are organized) and syntactic structures (rules for validating knowledge). This enables teachers to explain not just what is true, but why, how concepts interrelate, and how to address competing claims.
- *Pedagogical content knowledge (PCK)* combines content expertise with teaching skill, including effective ways to present concepts, select examples, and design explanations. It also involves anticipating student misconceptions and applying strategies to restructure understanding.
- *Curricular knowledge* is defined as familiarity with instructional materials, programs, and the broader curriculum. Teachers need lateral knowledge (what students are learning in other subjects) and vertical knowledge (what was and will be taught in the same subject) to integrate lessons effectively and select the most appropriate resources for diverse contexts.

In sum, effective teaching requires a complex, integrated knowledge base: deep subject expertise, pedagogical skill tailored to content, and an informed understanding of curriculum. Teacher preparation and assessment must reflect this complexity to ensure that educators are equipped not only to know their subject, but to teach it well.

Later research has shown that teachers' understanding of content directly influences their teaching methods (Ball et al., 2008). Critics point out that Shulman's model insufficiently accounts for the dynamic nature of teachers' knowledge and beliefs (Fennema & Franke, 1992; Petrou & Goulding, 2011). On this basis, numerous alternative models have emerged. Fennema and Franke (1992) proposed an interactive framework that includes four components: content knowledge, pedagogical knowledge, knowledge of student thinking, and teachers' beliefs. In this model, beliefs play an equal role to knowledge and are closely connected to teachers' practice.

Ball et al. (2008) expanded Shulman's framework into the Mathematical Knowledge for Teaching (MKT) model, based on empirical research on mathematics instruction. In the MKT model, Ball et al. (2008) divided mathematical knowledge into two main categories: common content knowledge (general mathematical knowledge) and specialized content knowledge (deep and teaching-specific knowledge). Specifically, MKT distinguishes between common content knowledge

(CCK), specialized content knowledge (SCK), and knowledge of content's horizon within CK, as well as knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of curriculum (KCC) within PCK. The model has significantly contributed to understanding the connections between teachers' knowledge and student achievement, but it has been criticized for not including the dimension of teachers' beliefs and for often having unclear boundaries between individual categories (Petrou & Goulding, 2011).

The most comprehensive approach is the Mathematics Teacher's Specialised Knowledge (MTSK) model, developed by Carrillo et al. (2018). The model is based on the analysis of classroom teaching, teacher training, and interviews, and introduces a new division of mathematical and didactic knowledge. Unlike earlier models, MTSK focuses specifically on the professional knowledge teachers need for teaching mathematics, rather than comparing it to knowledge from other professions. The MTSK model is divided into two main domains:

1. *Mathematical Knowledge (MK)* – knowledge of mathematics as a discipline, but in the context of teaching. This includes:
   - Knowledge of Topics (KoT): deep understanding of mathematical content, procedures, properties, representations, and applications.
   - Knowledge of the Structure of Mathematics (KSM): understanding how different areas of mathematics connect.
   - Knowledge of Practices in Mathematics (KPM): awareness of mathematical reasoning, problem-solving, and validation processes.
2. *Pedagogical Content Knowledge (PCK)* – knowledge of how to teach mathematics effectively. This includes:
   - Knowledge of Mathematics Teaching (KMT): strategies, resources, representations, and teaching approaches.
   - Knowledge of Features of Learning Mathematics (KFLM): understanding how students think, their strengths and weaknesses, emotional aspects, and ways they interact with content.
   - Knowledge of Mathematics Learning Standards (KMLS): awareness of curriculum expectations and standards.

A unique feature of the model is its emphasis on connecting mathematical knowledge and pedagogical content knowledge as two equal and complementary domains that jointly guide teachers' practice. At its core are teachers' beliefs about mathematics and its instruction, which both influence and are shaped by MK and PCK.

The common denominator of all the presented models is the recognition that successful mathematics teaching requires more than mastery of algorithms. Teachers must develop a complex combination of mathematical knowledge, didactic knowledge, and beliefs, which allows them to present content in multiple ways, respond to diverse student needs, and effectively address misconceptions (Shulman, 1986; Ball et al., 2008; Carrillo et al., 2018). Research also shows that the targeted development of didactic mathematical knowledge is often more effective than simply strengthening content knowledge (Zolfaghari, 2020).

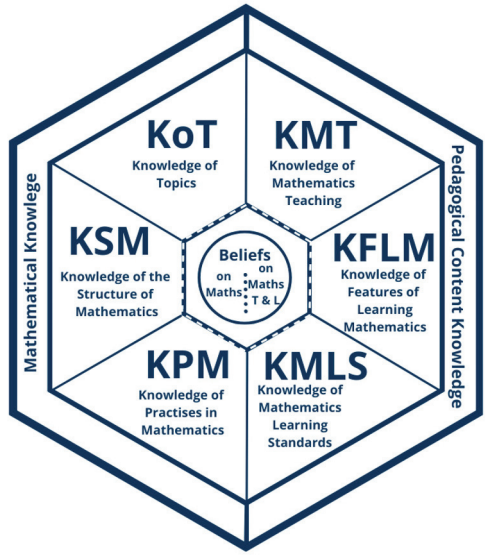

Figure 1. Model MTSK (adapted from Carrillo et al., 2018, p. 241).

While Shulman (1986), Ball et al. (2008) and Carrillo et al. (2018) systematically address teachers' didactic-professional knowledge, several authors (e.g., Fennema & Franke, 1992; Petrou & Goulding, 2011; Korthagen, 2004) point out the importance of including the dimension of personal beliefs and identity in order to understand teachers' professional development, as these deeper aspects significantly guide how teachers apply their knowledge in practice. Based on this, we will focus on the beliefs of future mathematics teachers about their pedagogical subject knowledge, which develops in parallel with the formation of subject-specific knowledge.

1.1. Metacognition, self-regulation and self-assessment

In addition to professional knowledge, modern higher education faces the challenge of fostering students' metacognitive awareness and integrating metacognitive knowledge into learning. Metacognition has become increasingly important in today's information-rich society, which demands adaptability and self-directed learning. Traditional academic skills are no longer sufficient; students must also develop metacognitive competencies to succeed (Balashov et al., 2021).

In the following the key concepts related to metacognition are defined: self-regulation, self-assessment, and self-reflection. We establish their interrelationships and present previous studies that have examined the influence of these processes on students' academic achievements.

Scholars such as Schraw and Moshman (1995), Pintrich (2002) and Schunk (2008) emphasize that metacognition includes both knowledge about cognition (understanding how one learns) and regulation of cognition (planning, monitoring, evaluating, and adjusting strategies). Vockell (2004, in Lew, Alwis &

Schmidt, 2010) describes metacognitive process as learners' awareness of their own knowledge and abilities, which helps them understand, monitor, and regulate their cognitive processes. Metacognitive knowledge refers to learners' knowledge of these processes and their ability to control and evaluate them. It consists of three main components: strategic knowledge (general learning and thinking strategies), task knowledge (knowing when to use a certain strategy), and self-knowledge (awareness of one's cognitive strengths and weaknesses as well as motivational beliefs about one's abilities, interests, and task value). When learners use such self-knowledge to make judgments about their performance, this is called self-assessment (Shrauger & Osberg, 1981). Similarly, Semana & Santos (2010) sees self-assessment as a metacognitive process in which individuals critically and consciously monitor their own thinking and actions. Clauss and Geedey (2010) add that metacognition includes predicting performance, managing one's knowledge, and identifying strengths and weaknesses in order to respond appropriately. According to McDonald (2007, 2011), self-assessment as part of metacognition becomes increasingly spontaneous, automatic, and in-depth as learners progress in their education.

Self-regulated learning is a process in which students set learning goals and actively monitor, control, and adjust their behavior, motivation, and knowledge to achieve them (Andrade, 2010; Pintrich, 2000). In doing so, they rely on both internal and external feedback (Andrade, 2010).

From a social-cognitive perspective, the most influential model of self-regulated learning is cyclical model (Schunk & Zimmerman, 1998), which includes three phases: forethought, performance and self-control, and self-reflection. According to Harris, Brown, and Harnett (2015), self-assessment plays a role in all three phases.

- Forethought involves preparation for learning: setting goals, forming beliefs about one's ability, and planning learning strategies.
- Performance and self-control occur during learning and include self-monitoring, where learners assess whether they are completing tasks correctly, often using criteria set by the teacher or by themselves.
- Self-reflection happens after learning, when learners evaluate their results against the criteria, consider which strategies were most effective, and reflect on the reasons for their success or difficulties.

Through this process, self-assessment influences all phases of self-regulated learning (Zimmerman & Moylan, 2009). Many researchers (Brown & Harris, 2014; Puustinen & Pulkkinen, 2001; Schunk, 2003; Tapia & Panadero, 2010) agree that self-assessment is one of the most important elements of self-regulation, as it involves awareness of goals and checking whether they have been achieved. Students who develop strong self-regulation skills can recognize what they know and what they do not know (Labuhn, Zimmerman & Hasselhorn, 2010).

Psychological studies since Flavell (1979) have shown that metacognition supports effective learning, self-regulation, and academic success. Encouraging students to consciously regulate their learning activities not only improves academic outcomes but also helps them recognize and address their own learning difficulties. Schoenfeld (2016) stresses the importance of creating an environment

that allows for interaction and confrontation of viewpoints, perspectives, ways of thinking. He defines self-regulation as one of the main broad areas covered by the term metacognition. According to Schoenfeld, monitoring, assessing progress and acting on progress assessments are key elements of self-regulation.

Research suggests that metacognitive awareness is a key educational resource, one that can and should be explicitly developed throughout higher education to enhance student learning and lifelong competence. Hattie (2009) conducted a meta-analysis of research examining the effect of different factors on students' academic achievement. He identified the development of metacognitive skills as one of the most important factors for improving students' achievement. Similarly, Dunlosky and Metcalfe (2009) emphasize that metacognition helps students understand their own thought processes and use this awareness to improve learning.

Lester et al. (1989) conducted a large-scale study to investigate the role of metacognition in mathematical problem-solving. They found out that there is a dynamic interaction between mathematical concepts and the metacognitive processes used to solve problems with these concepts. This means that control processes and awareness of cognitive processes develop in parallel with the understanding of mathematical concepts. Teaching problem-solving, and in particular teaching metacognition, is likely to be most effective if it is carried out in a systematically organised way under the guidance of a teacher.

More recently Dangin and Setyawan (2023) examined the relationship between metacognitive awareness, self-regulated learning, and academic achievement among senior university students in Indonesia. Their results show that both metacognitive awareness and self-regulated learning were significantly and positively correlated with academic achievement. Students with stronger skills in monitoring, planning, and evaluating their thinking tended to achieve higher academic achievement. Also, self-regulated learners – those who set goals, monitor progress, and adapt strategies – were more successful academically.

The study suggests that when combined, metacognitive awareness (how students think about and regulate learning) and self-regulated learning (how they act on that regulation) provide a more holistic understanding of student performance. Students who are metacognitively aware and self-regulated are better positioned to succeed academically. By cultivating these skills, educators can help learners take greater control over their learning, maintain motivation, and persist through difficulties.

2. Reflection

Reflection has long been recognized as a cornerstone of professional growth in education. While the concept appears simple – thinking about one's actions to improve practice – it has evolved into a rich and complex field of theory and application. Dewey laid the philosophical foundation of reflective thought. For him, reflection was "active, persistent, and careful consideration of any belief or supposed form of knowledge in the light of the grounds that support it and the

further conclusions to which it tends" (1933, p. 9). Dewey conceptualized reflection as a structured problem-solving process: identifying a problem, defining it, generating hypotheses, reasoning through them, and testing outcomes. Crucially, Dewey linked reflection with experience – learning is not merely doing but doing combined with critical thought. He also highlighted attitudes of open-mindedness, responsibility, and wholeheartedness as essential for genuine reflection.

Building on Dewey, Schön (1983) reframed reflection within the context of professional practice. He distinguished between three types of reflection:

— Reflection before action: it refers to all thought processes that occur before pedagogical action.

— Reflection-in-action occurs during the activity or teaching moment. It involves responding, adjusting, and experimenting in real time. Example: A teacher notices students are confused mid-lesson and decides to change the explanation method immediately. The key idea is that professionals should learn by doing, adapting their actions through immediate reflection.

— Reflection-on-action happens after the activity has ended. It involves looking back to analyze decisions, evaluate outcomes, and plan improvements. Example: After the lesson, the teacher reflects on why students struggled and how to improve next time. This type deepens understanding and contributes to professional growth over time.

This distinction helped teachers view reflection not merely as an afterthought, but as a process that can guide their actions in real time. It enabled a shift from teachers as practitioners to teachers as (self-) critical professionals – those who think through their actions in advance, remain aware of the consequences of their pedagogical decisions, and critically evaluate or reevaluate them afterward. In this way, reflection became a means of bridging theory and practice, fostering adaptive expertise.

Korthagen (2010) developed an understanding of reflection as a tool for professional development. According to him, reflection is not merely an analytical evaluation of one's own practice; it is also a process that enables teachers to connect theory and practice, recognise patterns in their work and develop a deeper awareness of their teaching identity. It is important for beginners to initially focus on lesson implementation and student responses before moving on to deeper layers, such as beliefs and identity.

Research has often used reflective writing as a tool to scaffold reflection. Cohen-Sayag and Fischl (2012) showed that structured reflective writing can help pre-service teachers develop deeper thinking, though progress depends heavily on the kind of feedback they receive. Lane et al. (2014) proposed distinctions between purely descriptive writing and higher-level reflective texts, underscoring the need for explicit guidance. More recent research sheds light on issues that remain underexplored in the field of teacher reflection. Páez et al. (2020) call for bridging the gap between theoretical teacher education and the day-to-day realities of classrooms, particularly in under-resourced contexts. Similarly, Olawale and Hendricks (2024) emphasize that even teachers with strong subject knowledge may fall short if their sense of self-efficacy in teaching practices is weak.

Taken together, these studies reveal a recurring research gap: the emphasis tends to fall on *what* teachers should reflect on, while far less attention is given to *how* reflection should be practiced. Marcos, Sánchez, and Tillema (2011) likewise noted that much of the literature describes reflection in idealized terms but offers little practical guidance. Fox, Dodman, and Holincheck (2019) go further, warning that reflection can become a "hall of mirrors," reinforcing existing beliefs rather than challenging them.

These critiques highlight an enduring tension between the vision of teachers as critically transformative agents and the practical realities of implementation – time, training, feedback, and institutional constraints. Reflection is therefore not a static ideal but a dynamic, contested practice shaped by philosophy, pedagogy, and policy. For it to reach its transformative potential, teacher education must move beyond prescriptive ideals toward approaches that are both critically engaged and context-sensitive.

3. Research focus

The present study investigates how mathematics students self-assess their competences and progress at the end of teaching practice, and how they reflect on their teaching practice after mathematics lessons. Rather than directly measuring their MKT, the study explores students' metacognitive awareness of their own PCK, drawing on Carrillo et al.'s (2018) framework.

Self-assessment and reflection focus on three aspects of PCK:

— Knowledge of Mathematics Teaching (KMT): self-assessment of class management (instructions, motivating students, classroom leadership, communication with students, performance and speech) and of the teaching and learning process (learning environment, clarity of goals, methods and strategies, monitoring, content accuracy, representations, problem-solving).

— Knowledge of Mathematics Learning Standards (KMLS): self-assessment of lesson planning (lesson structure, articulation of goals, selection of content and activities, monitoring students, differentiation, teaching methods, use of resources).

— Knowledge of Features of Learning Mathematics (KFLM): reflection on how students think, their strengths and weaknesses, emotional aspects, and ways they interact with content.

4. Research questions

This study addresses the following research questions:

From the field of self-assessment of mathematics students' competences:

1. How do students assess their teaching competences?

2. Are there any discrepancies in students' self-assessment of competences between the 3rd and 4th year?
3. In which competences do students assess the greatest progress in teaching?

From the field of reflection on mathematics students' own teaching practice:

4. Which aspects of their own teaching practice do mathematics students perceive as important in the self-reflection of their lessons?
5. Which aspects of their own teaching practice do students perceive as their strengths and which as their weaknesses?

5. Method

The empirical study employed descriptive, causal, and non-experimental methods of pedagogical research (Hartas, 2010; Sagadin, 1991). It was designed as a mixed-methods study, integrating both qualitative and quantitative approaches. The quantitative component was used to analyze students' self-assessment of competences and to provide answers to the research questions, whereas the qualitative component focused on the content analysis and interpretation of students' written reflections.

5.1. Sample

The study was conducted at the Faculty of Education, University of Ljubljana, Slovenia, during two consecutive academic years (2023/24 and 2024/25). The sample included 23 mathematics students (two-subject mathematics program), observed during their 3rd and 4th years of study.

5.2. Data collection and measuring instruments

During their teaching practice, students kept a teaching practice diary. The diary provided a structured template for preparing lesson plans, space for written reflections after each lesson, and a self-assessment questionnaire to evaluate their own competences. Data were drawn from:

— a questionnaire on self-assessment of teaching competences at the end of teaching practice (for 1st, 2nd and 3rd research question)
— written reflections after each mathematics lesson (for 4th and 5th research question)

This allowed insight into both self-assessment of competences and reflection on teaching practice.

The questionnaire on self-assessment of teaching competences was structured around three main categories: lesson planning, classroom management, and the teaching and learning process.

The first category focused on preparation before conducting a lesson, while the other two addressed the implementation of teaching, specifically pedagogical–interactional aspects (class management) and didactical–content-related aspects of mathematics instruction (teaching and learning process). Category class management referred to how teachers establish and manage interactions with students and shape the learning environment. The teaching and learning process, on the other hand, concerned the teacher's subject-specific decisions in the teaching of mathematics. In this sense, the two categories differed in emphasis: the first highlighted general didactical competences, whereas the second emphasized mathematics-specific didactical competences. Each category was further divided into subcategories, which are presented in Tables 1, 2, and 3.

Table 1. A questionnaire on self-assessment of teaching competences at the end of teaching practice – Part 1: Lesson planning.

01 Carrying out the preparation of the lesson plan The lesson plan is presented to the mentor on time and contains all the necessary components (formal data, learning objectives, content development, teaching methods, student activities, learning styles, resources, tools).	
Notes by the evaluator	
02 Articulating learning objectives. The student formulates appropriate operational objectives that are relevant and clearly related to the planned content and activities of the lesson and selects appropriate teaching methods and learning styles for them.	
Notes by the evaluator	
03 Planning learning content and activities. When planning content and activities, the student takes into account their relevance and interest for the learners, and the links with other units and subjects are indicated.	
Notes by the evaluator	
04 Planning students' monitoring. The student shall make provision for appropriate ways of monitoring and checking the achievement of the learning objectives by the learners.	
Notes by the evaluator	
05 Planning teaching differentiation. The student plans possible adaptations for different learners needs, especially appropriate forms of internal differentiation.	
Notes by the evaluator	
06 Planning teaching methods and learning styles. The plan is to intervene with different teaching methods. The lesson is sufficiently varied in this respect.	
Notes by the evaluator	
07 Planning teaching resources, teaching materials. The planned teaching resources and teaching materials are appropriately selected and/or designed (according to content and objectives).	
Notes by the evaluator	

Table 2. A questionnaire on self-assessment of teaching competences at the end of teaching practice – Part 2: Class management.

11 Giving instructions. The instructions given are clear, brief and clearly communicate what is expected of the learners.	
Notes by the evaluator	
12 Motivating students. Emphasizes the importance of learning in each activity, provides opportunities to achieve objectives when appropriate work is done, uses rewards and praise appropriately, comments on reasons for achievement and non-achievement of objectives appropriately.	
Notes by the evaluator	
13 Organization of the learning environment. The learning environment is well prepared and organised, allowing for a good working atmosphere in the classroom (no time wasting, discipline problems).	
Notes by the evaluator	
14 Communication with students. Establishes appropriate contact (spoken and non-spoken), takes all pupils into account (even the more passive ones), actually listens to them, encourages them, responds to their suggestions and gives them enough time to respond, moves appropriately around the classroom.	
Notes by the evaluator	
15 Performance and speech. Speaks correctly, clearly, naturally, understandably, in a way that engages the child, emphasizes the importance.	
Notes by the evaluator	

Table 3. A questionnaire on self-assessment of teaching competences at the end of teaching practice – Part 3: Teaching and learning process.

21 Clarity of lesson objectives. The implementation of the lesson makes the learning objectives clear, presents expectations appropriately, activates the necessary background knowledge appropriately at the beginning, and concludes and summarizes the lesson appropriately.	
Notes by the evaluator	
22 Teaching methods and learning styles. Teaching methods and learning styles are clearly evident and appropriately guided or implemented.	
Notes by the evaluator	
23 Monitoring students' progress. Monitors students' work at all times, correcting errors, providing ample opportunities for consolidation and allowing students to revisit content through guided practice.	
Notes by the evaluator	
24 Relevance of the presentation stage of the lesson. The presentation of the mathematical content is appropriately challenging, mathematically correct, focused on the essentials, the formulations of the definitions are clear, precise and adequately commented, supported by examples and appropriate exercises.	
Notes by the evaluator	
25 Examples and representations. Use of appropriate concrete, pictorial and symbolic representations, illustration of concepts and procedures with appropriate examples, appropriate gradation of complexity of examples.	
Notes by the evaluator	
26 Problem-solving skills. Integrates process skills (documenting, analyzing, validating, reasoning, etc.) and problem-solving strategies appropriately.	
Notes by the evaluator	

In addressing the first, second, and third research questions, we applied closed coding, as the possible answers had been pre-determined and students could select among them when completing the self-assessment rubrics in the questionnaire. They could choose from the following options:

A. I have been good at this aspect all along
B. I have seen a lot of progress in myself
C. I have seen some progress in myself
D. I still feel insecure

The self-assessment categories are arranged hierarchically. The level of self-assessment of competence attainment decreases from A to D.

In the first RQ, the focus was on the self-assessment of competences that students consider themselves to have fully achieved (already mastered before teaching practice). The following self-assessment categories were analyzed:

1. Mastered in the 3rd year: Students scored an A in the third year.
2. Mastered in the 4th year: Students scored an A in the fourth year.
3. Mastered in both years: Students scored an A in both years (A-A).

Two categories were formed to identify discrepancies in self-assessment of competences in the 3rd and 4th years (2nd RQ).

1. Positive discrepancies: Students scored higher in their self-assessment in the 4th year than in the 3rd year (B-A, C-A, D-A, D-B, D-C).
2. Negative discrepancies: Students scored lower in their self-assessment in the 4th year than in the 3rd year (A-B, A-C, A-D, B-D, C-D).

In order to determine the progress of the students during the teaching practice (3rd RQ), we formed two categories:

1. Mastered by the end of the 4th year: Students scored B, C or D in the 3rd year and A in the 4th year.
2. Progress in teaching practice: Students scored progress in both years of teaching practice (B-B, B-C, C-B, C-C).

The template for written lesson reflection guided students to identify their strengths and weaknesses and to suggest improvements to their teaching. To support this process, students were provided with the following questions:

1. Which aspects of the lesson were you particularly satisfied with?
2. In which ways, and how, would you improve the lesson?

Each student's practice diary contained 6–8 different mathematics lesson plans accompanied by reflections on their implementation. In total, we analyzed 343 written reflections, 174 from the 3rd year and 169 from the 4th year.

To identify (important) aspects of students' reflection on their own teaching practice (4th and 5th research question) we applied an inductive approach of open coding to the responses to questions 1 and 2 in the reflection reports. This meant that codes were generated during the analysis and subsequently grouped into broader

categories, using thematic coding. The formation of categories and subcategories of aspects of students' reflection on their own teaching practice is presented in the Results section.

In the following, we briefly present the competences that students – future mathematics teachers acquire during their mathematics studies, both in the field of PCK and in the areas of metacognition and self-reflection.

5.3. Mathematics teacher study program

The two-subject mathematics degree program is a first-cycle study track that prepares students to become teachers of mathematics combined with a second subject (physics, computer science, or technology). Teaching practice takes place in the 3rd and 4th year, each lasting two weeks annually (one week in mathematics, one in the second subject). Students are required to independently conduct mathematics lessons and engage in teaching practice under supervision.

The mathematics teacher program combines theoretical knowledge, didactical competences, and practical teaching experience. It includes three didactical courses, each aligned with the progressive development of teaching competences:

1. *2nd year: The Fundamentals of the Didactics of Mathematics* Students gain a foundational understanding of mathematical knowledge, the organization of learning processes, problem-solving, communication in mathematics, use of textbooks and technology, and approaches for working with both gifted students and those with learning difficulties.
2. *3rd year: The Didactics of Primary School Mathematics* Focus shifts to integrating mathematical, pedagogical, and educational knowledge in classroom contexts. Students develop skills to design and adapt curricula, select appropriate teaching methods, reflect on their own teaching, and account for diverse learning styles in mathematics.
3. *4th year: Topics from the Didactics of Mathematics* Emphasis is placed on advanced competences such as curriculum development, assessment and feedback, interdisciplinary collaboration, use of ICT in teaching, and continuous self-reflection to improve practice.

Across the 3rd and 4th years, *reflection on teaching practice* is systematically developed. After conducting lessons, students write a structured analysis, focusing on anticipated and actual learning situations, their teaching decisions, outcomes, and improvements. These reflections must also be supported by theoretical knowledge from mathematics education and psychology.

By the end of their studies, students are expected to demonstrate subject expertise, the ability to design and deliver effective mathematics instruction, and the metacognitive capacity to critically reflect on and continuously improve their teaching practice.

6. Results

The results of the study are presented in two parts. The first part refers to the students' self-assessment of their competences at the end of their teaching practice. The second part refers to the reflection on students' own teaching practice (the importance of the competences and the reflection on strengths and weaknesses in teaching after lessons have been conducted).

6.1. Students' self-assessment of their competences at the end of their teaching practice

The first research question aimed to determine how students self-assess their acquisition of competences for teaching mathematics at the end of their teaching practice. The competences were divided into three categories: Seven competences in lesson planning (Table 1), five competences in classroom management (Table 2) and six competences in teaching and learning process (Table 3).

Based on an analysis of students' self-assessments of their acquisition of competences in the category of lesson planning (Table 4), we found that in both the third and the fourth year, three out of seven competences stand out, with more than half of the students believing that they had already mastered them before the teaching practice. In the third year, these competences were: carrying out the preparation of the lesson plan, planning learning content and activities, and planning teaching resources and materials. In the fourth year, students assessed their competences similarly, except that, rather than mastering the competence of planning learning acquisition, a larger proportion of students believed that they had already mastered the competence of articulating learning objectives before their teaching practice.

Table 4. Students' self-assessment of their acquisition of lesson planning competences.

Lesson planning – mastered competences						
	Mastered in the 3rd year		**Mastered in the 4th year**		**Mastered in both years**	
	f	*f %*	*f*	*f %*	*f*	*f %*
Carrying out the preparation of the lesson plan	**12**	**52.2**	**12**	**52.2**	7	30.4
Articulating learning objectives	11	47.8	**17**	**73.9**	10	43.5
Planning learning content and activities	**12**	**52.2**	10	43.5	6	26.1
Planning student monitoring	9	39.1	10	43.5	6	26.1
Planning teaching differentiation	4	17.4	5	21.7	2	8.7
Planning teaching methods and learning styles	9	39.1	8	34.8	5	21.7
Planning teaching resources, teaching materials	**17**	**73.9**	**17**	**73.9**	**16**	**69.6**

The largest proportion of students who assessed that they had mastered a certain competence in both years was in planning teaching resources and materials (69.6%), while the smallest proportion was in planning teaching differentiation (8.7%).

The proportion of students who assessed that they had mastered the particular competence in lesson planning was similar in the third and fourth year. The biggest differences between the third and fourth year in self-assessment of competences are evident in the competence of articulating learning objectives. The proportion of students who self-assessed that they had mastered this competence was higher in the fourth year (73.9%) than in the third year (47.8%).

With regard to classroom management competences (Table 5), over half of the students in both their third and fourth years said that they had already mastered three out of five competences before the teaching practice. The largest proportion of students who assessed that they had already mastered a competence in both years were in two subcategories of competences: motivating students (47.8%) and communication with students (60.9%). In addition to these two competences, more than half of third-year students assessed that they had already mastered the competence of giving instructions, whereas fourth-year students assessed that they had already mastered the competence of organising the learning environment.

Table 5. Students' self-assessment of their acquisition of classroom management competences.

Classroom management – mastered competences						
	Mastered in the 3rd year		**Mastered in the 4th year**		**Mastered in both years**	
	f	*f %*	*f*	*f %*	*f*	*f %*
Giving instructions	**12**	**52.2**	10	43.5	6	26.1
Motivating students	**13**	**56.5**	**13**	**56.5**	**11**	**47.8**
Organization of the learning environment	6	26.1	**12**	**52.2**	4	17.4
Communication with students	**15**	**65.2**	**18**	**78.3**	**14**	**60.9**
Performance and speech	9	39.1	9	39.1	6	26.1

The proportion of students who assessed that they had mastered the individual competence in the category classroom management was similar in the third and fourth year. The greatest differences between the third and fourth years are evident in the competence of organising the learning environment. The proportion of students who self-assessed that they had mastered this competence was higher in the fourth year (52.2%) than in the third year (26.1%). Organising learning environment is also the competence that the smallest percentage of students in both years assessed as one that they had mastered before the teaching practice.

With regard to the teaching and learning process competences (Table 6), over half of the students in both their third and fourth years assessed that they had already mastered four out of six competences. The largest proportion of students who

assessed that they had mastered a competence was in the competence of monitoring students' progress (60.9%), while the smallest proportion was in the competence of problem-solving skills (8.7%). In the third year, more than half of the students assessed that they had already mastered three competences: teaching methods and learning styles, the relevance of the presentation stage of the lesson and the use of examples and representations. In the fourth year, however, more than half of the students assessed that they had mastered clarity of lesson objectives instead.

The proportion of students who assessed that they had mastered the individual competence in the category teaching and learning process was similar in the third and fourth year. The greatest differences in self-assessment of competences between the third and fourth years are evident in two competences: clarity of lesson objectives and examples and representations. A higher proportion of students self-assessed the competence of acquisition of the clarity of lesson objectives in the fourth year (60.9%) than in the third year (39.1%). Conversely, the proportion of students who self-assessed the acquisition of the competence examples and representations was higher in the third year (69.6%) than in the fourth year (39.1%).

Table 6. Students' self-assessment of their acquisition of teaching and learning process competences.

Teaching and learning process – mastered competences						
	Mastered in the 3rd year		**Mastered in the 4th year**		**Mastered in both years**	
	f	*f %*	*f*	*f %*	*f*	*f %*
Clarity of lesson objectives	9	39.1	**14**	**60.9**	7	30.4
Teaching methods and learning styles	**13**	**56.5**	**12**	**52.2**	9	39.1
Monitoring students' progress	**16**	**69.6**	**19**	**82.6**	**14**	**60.9**
Relevance of the presentation stage of the lesson	**14**	**60.9**	**12**	**52.2**	8	34.8
Examples and representations	**16**	**69.6**	9	39.1	7	30.4
Problem-solving skills	6	26.1	4	17.4	2	8.7

With the second research question, we wanted to find out whether there are any discrepancies in students' self-assessments of competences in the third and fourth years, and if so, what these discrepancies might be. In contrast to RQ 1, here we analyzed the results from the perspective of changes in the individuals. To this end, we formed two categories of discrepancies. The first category consists of positive discrepancies, where students in their third year assess their individual competences at a lower level than in their fourth year. Positive discrepancies therefore indicate individual progress during teaching practice or the acquisition of competences during the transition from the third to the fourth year. The second group consists of negative discrepancies, which indicate that students in their third year assess themselves higher than in their fourth year. This usually meant that in the third year the student assessed that they had already mastered the competence, whereas in the fourth year they assessed that they had made progress in this competence. Discrepancies between the assessments were not taken into account in cases

where students indicated that they had already mastered a certain competence in both years or showed progress in developing a particular competence in both years.

Table 7 shows the proportion of students who had positive or negative discrepancies in their self-assessment of competences at the end of their teaching practice after the fourth year of study for the category of lesson planning competences. The largest proportion of students with positive discrepancies was in the self-assessment of two competences, namely the competence of articulating learning objectives (34.8%) and the competence of carrying out the preparation of the lesson plan (26.1%). Negative discrepancies were observed in almost a third of the students for the competence of planning learning content and activities (30.4%).

Table 7. Discrepancies in students' self-assessment of lesson planning competences.

Lesson planning: self-assessment discrepancies				
	Positive discrepancies		**Negative discrepancies**	
	f	*f %*	*f*	*f %*
Carrying out the preparation of the lesson plan	**6**	**26.1**	5	21.7
Articulating learning objectives	**8**	**34.8**	1	4.3
Planning learning content and activities	4	17.4	**7**	**30.4**
Planning student monitoring	5	21.7	3	13.0
Planning teaching differentiation	4	17.4	3	13.0
Planning teaching methods and learning styles	2	8.7	4	17.4
Planning teaching resources, teaching materials	1	4.3	1	4.3

Table 8 shows that, in the category of classroom management, there are mostly no discrepancies between students' self-assessments of their competences in the third and fourth years. Approximately one-third of students showed positive discrepancies in the competence of organization of the learning environment (34.8%) and negative discrepancies in the competence of giving instructions (26.1%).

Table 8. Discrepancies in students' self-assessment of classroom management competences.

Classroom management: self-assessment discrepancies				
	Positive discrepancies		**Negative discrepancies**	
	f	*f %*	*f*	*f %*
Giving instructions	4	17.4	**6**	**26.1**
Motivating students	2	8.7	2	8.7
Organization of the learning environment	**8**	**34.8**	2	8.7
Communication with students	4	17.4	1	4.3
Performance and speech	4	17.4	3	13.0

Table 9 shows the proportion of students who had positive or negative discrepancies in their self-assessment of their teaching and learning process competences. Around one-third of students had positive discrepancies in their self-assessment of two competences: clarity of lesson objectives (30.4%) and problem-solving skills (26.1%). Similarly, one-third of students had negative discrepancies in three out of six competencies: the relevance of the presentation stage of the lesson (26.1%), examples and representations (39.1%), and problem-solving skills (26.1%).

Table 9. Discrepancies in students' self-assessment of teaching and learning process competences.

Teaching and learning process: self-assessment discrepancies				
	Positive discrepancies		**Negative discrepancies**	
	f	*f %*	*f*	*f %*
Clarity of lesson objectives	**7**	**30.4**	2	8.7
Teaching methods and learning styles	4	17.4	4	17.4
Monitoring students' progress	5	21.7	2	8.7
Relevance of the presentation stage of the lesson	4	17.4	**6**	**26.1**
Examples and representations	3	13.0	**9**	**39.1**
Problem-solving skills	**6**	**26.1**	**6**	**26.1**

The third research question addressed the competencies in which students perceived progress during their teaching practice. In the category of lesson planning competences (Table 10), the largest proportion of students (78.3%) assessed that they had made progress (total – progress) in the competence of planning teaching differentiation. However, this is also a competence that only a small proportion of students (13.0%) had mastered by the end of their teaching practice in the fourth year. The competences in which more than half of students assessed progress

Table 10. Proportion of students who made progress in lesson planning competences during their teaching practice.

Lesson planning – progress						
	Mastered by the end of the teaching practice		**Progress – not mastered**		**Total – progress**	
	f	*f %*	*f*	*f %*	*f*	*f %*
Carrying out the preparation of the lesson plan	5	21.7	6	26.1	11	47.8
Articulating learning objectives	**7**	**30.4**	5	21.7	**12**	**52.2**
Planning learning content and activities	4	17.4	6	26.1	10	43.5
Planning students' monitoring	4	17.4	10	43.5	**14**	**60.9**
Planning teaching differentiation	3	13.0	**15**	**65.2**	**18**	**78.3**
Planning teaching methods and learning styles	2	8.7	11	47.8	**13**	**56.5**
Planning teaching resources, teaching materials	1	4.3	5	21.7	6	26.1

during their teaching practice were also: planning students' monitoring (60.9%), planning teaching methods and learning styles (56.5%), and articulating learning objectives (52.2%). The largest proportion of students (30.4%) assessed that they had acquired the latter competence by the end of their teaching practice. The lowest proportion of students assessed progress in the competence of planning teaching resources and materials (26.1%).

With regard to classroom management competences (Table 11), the largest proportion of students assessed progress in the competence of organization of the learning environment (73.9%) and in the competence of performance and speech (60.9%). Fewer than half of all students made progress in other competences. The smallest proportion of students made progress in the competence communicating with students (34.8%). However, for this competence, the majority of students assessed that they had mastered it before teaching practice (Table 5). In the organization of the learning environment competence, the largest proportion of students (34.8%) assessed that they had made progress and mastered it by the end of their teaching practice. Meanwhile, the largest proportion of students made progress in the competence of performance and speech in both years, but did not master it (47.8%).

Table 11. Proportion of students who made progress in classroom management competences during their teaching practice.

Classroom management – progress						
	Mastered by the end of the teaching practice		**Progress – not mastered**		**Total – progress**	
	f	*f %*	*f*	*f %*	*f*	*f %*
Giving instructions	4	17.4	7	30.4	11	47.8
Motivating students	2	8.7	8	34.8	10	43.5
Organization of the learning environment	**8**	**34.8**	9	39.1	**17**	**73.9**
Communication with students	4	17.4	4	17.4	8	34.8
Performance and speech	3	13.0	**11**	**47.8**	**14**	**60.9**

In the teaching and learning process category (Table 12), the largest proportion of students assessed that they had made progress in the problem-solving skills and clarity of lesson objectives competences during their teaching practice (total – progress). Fewer than half of all students assessed progress in the other competences in this category. A large proportion of students assessed that they had made progress in problem-solving skills (56.5%), but not mastered them (progress – not mastered), while a very small proportion assessed that they had mastered them by the end of their teaching practice (8.7%). The largest proportion of students (30.4%) assessed that they had made progress in the competence of clarity of the learning lesson and had mastered it by the end of their teaching practice.

Table 12. Proportion of students who made progress in teaching and learning process competences during their teaching practice.

Teaching and learning process – progress						
	Mastered by the end of the teaching practice		**Progress – not mastered**		**Total – progress**	
	f	*f %*	*f*	*f %*	*f*	*f %*
Clarity of lesson objectives	**7**	**30.4**	7	30.4	**14**	**60.9**
Teaching methods and learning styles	3	13.0	7	30.4	10	43.5
Monitoring students' progress	5	21.7	2	8.7	7	30.4
Relevance of the presentation stage of the lesson	4	17.4	5	21.7	9	39.1
Examples and representations	2	8.7	5	21.7	7	30.4
Problem-solving skills	2	8.7	**13**	**56.5**	**15**	**65.2**

6.2. Reflection on mathematics students' own teaching practice

In this section, we will focus on the last two research questions, namely, which aspects of teaching are important to students and are analyzed in their reflections on lessons, and which of these are analyzed as the student's strengths and which ones as the student's weak areas of knowledge for teaching mathematics. In analyzing the importance of the aspects, we took into account the analyses of all student reflections on the lessons conducted in both the 3rd and 4th years.

Based on a content analysis of the students' reflections on the lessons, we formed three main categories of teacher knowledge for teaching mathematics, which coincide with the categories of competence in the self-assessment sheet: lesson planning, classroom management, teaching and learning process. Within these categories, we identified the same competences as those listed by students in the self-assessment sheet. The only competence that did not appear in the reflections at the end of the lesson was the competence of carrying out the preparation of the lesson plan.

Table 13 shows the frequency of individual competences addressed by students in their reflections, as well as whether they addressed this competence as their strong or weak area of knowledge. We combined the data for students in the 3rd and 4th years.

As shown in Table 13, when reflecting on the lessons, students most frequently addressed competences related to lesson implementation (83.1%), with around half (49.3%) of the competences reflected upon being from the category of teaching and learning process, and around one-third (33.8%) from the category of classroom management. Students rarely addressed competences from the category of lesson planning in their reflections.

Table 13. Frequency of competences addressed by students in their reflections.

Self-reflection competences	**Strengths**		**Weaknesses**		**Total**	
	f	*f %*	*f*	*f %*	*f*	*f %*
Articulating learning objectives	0	0.0	1	0.3	1	0.1
Planning learning content and activities	15	2.6	8	2.2	23	2.5
Planning students' monitoring	0	0.0	1	0.3	1	0.1
Planning teaching differentiation	22	3.9	49	**13.7**	71	**7.6**
Planning teaching methods and learning styles	10	1.8	9	2.5	19	2.0
Planning teaching resources, teaching materials	27	**4.7**	15	4.2	42	4.5
Total: Lesson planning	74	**13.0**	83	**23.2**	157	**16.9**
Giving instructions	8	1.4	18	5.0	26	2.8
Motivating students	69	**12.1**	28	7.8	97	**10.4**
Organization of the learning environment	41	7.2	30	**8.4**	71	7.6
Communication with students	75	**13.1**	28	7.8	103	**11.1**
Performance and speech	12	2.1	5	1.4	17	1.8
Total: Classroom management	205	**35.9**	109	**30.4**	314	**33.8**
Clarity of lesson objectives	27	4.7	7	2.0	34	3.7
Teaching methods and learning styles	39	6.8	39	10.9	78	8.4
Monitoring students' progress	80	**14.0**	30	8.4	110	**11.8**
Relevance of the presentation stage of the lesson	75	**13.1**	45	**12.6**	120	**12.9**
Examples and representations	58	10.2	42	**11.7**	100	**10.8**
Problem-solving skills	13	2.3	3	0.8	16	1.7
Total: Teaching and learning process	292	**51.1**	166	**46.4**	458	**49.3**

In the lesson planning category, the competence that was addressed most frequently was planning teaching differentiation (7.6%). This competence was most often identified as an area in which students needed improvement (13.7%). Planning teaching resources and materials was the most frequently addressed competence as students' strengths in this category (4.7%). Students have expressed satisfaction with the designed teaching materials and worksheets.

Among classroom management competences, the competences of communicating with students (11.1%) and motivating them (10.4%) were the most often reflected on by students. These two competences were most often reflected upon as areas of strength (communication: 13.1%, motivation: 12.1%). As areas for improvement, students most often reflected on the competence of organising of the learning environment (8.4%), focusing on creating an appropriate working atmosphere in the classroom and establishing discipline. Students also frequently mentioned communication with students (7.8%) and motivating them (7.8%) as areas for improvement.

When it comes to teaching and learning process competences, three aspects stand out in terms of how often they are reflected on: relevance of the presentation

stage of the lesson (12.9%), monitoring students' progress (11.8%), and examples and representations (10.8%). The relevance of the presentation stage of the lesson was the aspect most frequently considered as both a strength (13.1%) and a weakness (12.6%). Students therefore consider this competence from the perspectives of both opportunities for improvement and effective practice. The competence of monitoring students' progress was the most frequently highlighted area of strength (14.0%). In addition to the competence of the relevance of the presentation stage of the lesson, students most often reflected on the competence of examples and representations as a weak area of knowledge. The most frequently mentioned issues in this competence were insufficient clarity, not writing systematically on the blackboard and not using models to represent mathematical concepts. Two teaching and learning competences that we emphasize as important for quality teaching but which were rarely mentioned in student reflections are worth noting: clarity of lesson objectives (3.7%) and problem-solving skills (1.7%).

7. Discussion

We present the findings related to both research objectives and offer their critical justification in light of previous international studies in this field.

Through the first objective of our research, we gained insight into students' self-assessment of their teaching competences at the end of their teaching practice. Specifically, we examined which competences students felt they had mastered, where the largest discrepancies appeared in self-assessment between the third and fourth year, and which competences students most often attributed to progress during their teaching practice. In each competence area, we identified at least one competence that more than half of the students in both years self-assessed having already mastered. This suggests that students do not attribute these competences to pedagogical training, but rather to prior experiences and knowledge acquired during their studies. These included planning teaching resources and materials, communicating with students, motivating them, and monitoring their progress. By contrast, the competences that the fewest students felt they had mastered were planning teaching differentiation and problem-solving skills. Although these areas are strongly emphasized in the study program, students indicated that they had not yet mastered them and required more experience to do so.

When comparing third- and fourth-year self-assessments, we found relatively few differences overall. The clearest distinctions were in articulating learning objectives, organizing the learning environment, and ensuring the clarity of lesson objectives – competences which a greater proportion of fourth-year students rated as mastered compared to third-year students. These findings are encouraging, as we would expect final-year students to rate themselves higher in these areas. These competences relate both to lesson planning and classroom management, the latter being a skill that develops primarily through practice. Interestingly, the competence with the largest difference in the opposite direction was the use of examples and representations. A possible explanation may lie either in the greater self-criticism of older students or in differences in the content taught.

We also explored the extent of discrepancies in self-assessments between third- and fourth-year students. For most students, self-assessments were fairly consistent across the two years, in the sense that they either indicated steady progress without mastery, or consistently rated themselves as having already mastered the competence. When discrepancies did occur, we were especially interested in positive discrepancies, which signal progress from the third to the fourth year. These were found in about one-third of students for competences such as preparing lesson plans, articulating objectives, organizing the learning environment, ensuring clarity of objectives, and problem-solving skills. Less desirable were negative discrepancies, where students gave lower ratings in the fourth year compared to the third. These appeared in about one-third of students for competences such as planning learning content and activities, giving instructions, the relevance of the presentation stage of the lesson, use of examples and representations, and problem-solving skills. Most of these competences fall under the category of teaching and learning processes. We believe negative discrepancies may result from two factors: the first is related to experiential learning – students base their self-assessment on prior experiences gained during teaching practice, which may have been different in the third year compared to the fourth year (e.g., teaching different math content). The second factor is related to the students' growing knowledge and self-critical awareness, which may lead them to assess themselves more rigorously in the fourth year.

The third research question focused on identifying competences where students most often perceived progress during teaching practice. Most of these competences were related to lesson planning, including planning for differentiation, monitoring student progress, choosing teaching methods and learning styles, and articulating learning objectives. Differentiation stands out in particular: although only a small proportion of students considered it mastered, many reported making progress in this area during their teaching practice. This is especially important, as Valenčič Zuljan et al. (2007) stress that teachers must be able to adapt lessons for lower-achieving students and track their progress. In classroom management, the most frequently reported areas of progress were organizing the learning environment and developing performance and speech skills – competences that are closely tied to experience, so improvement here is expected. In the domain of teaching and learning processes, the greatest reported progress was in clarity of lesson objectives, problem-solving skills, and teaching methods. Again, problem-solving stands out as a competence that only a few students had fully mastered, yet many reported significant progress in this area during their teaching practice. This finding is particularly important, as previous studies with student teachers have shown that the frequency of including problem-solving tasks in mathematics lessons is closely linked to teachers' own problem-solving competences (Manfreda Kolar & Hodnik, 2011).

The second objective of the study was to explore which aspects of their teaching practice students emphasized in their reflections. The analysis revealed areas and sub-areas of competence that students perceived as strengths or weaknesses – an element that, according to Clauss and Geedey (2010), represents one of the three key dimensions of an individual's metacognitive awareness. Notably, the aspects most often reflected upon correspond closely to the competences included

in the self-assessment sheets. This alignment reflects Shulman's (1986) concept of pedagogical content knowledge (PCK), which integrates content knowledge, didactics, and curricular knowledge. In their reflections, students address all the important competences emphasized during their studies and suggested as a basis for self-assessment of their progress during teaching practice. However, we found that the ratio of competences reflected upon by students is unequal. The smallest proportion of reflections relates to competences in the area of lesson planning. This is consistent with Korthagen (2010), who emphasizes that beginners often have a more formal understanding of lesson planning and pay more attention to direct experiences of implementation and student responses. Students address competences in the field of teaching and learning processes the most in their reflections. We identified this in approximately half of all lessons reflected upon, indicating that students attribute the greatest importance to it. Within this area of competence, students most often emphasized the importance of providing examples and representations and ensuring content relevance. This supports the findings of Ball et al. (2008), who argue that the quality of mathematics teaching is primarily determined by the teacher's ability to explain and present content in different ways and anticipate students' misconceptions. Among the competences in the field of classroom management motivating and communicating with students were most often emphasized, which aligns with Hattie's (2009) findings on the importance of teachers' social-emotional skills for student achievement.

As for strengths, students most often identified monitoring student progress, ensuring relevance in the presentation phase, communicating, and motivating students – competences that are strongly supported during their program of study. These self-perceptions are consistent with their self-assessment ratings, where the majority indicated mastery of these same competences throughout their teaching practice. By contrast, weaker areas most often included planning teaching differentiation, using examples and representations, and ensuring relevance of the presentation stage of the lesson. Planning teaching differentiation, in particular, was consistently identified as challenging both in reflections and self-assessments.

Of particular note is again the competence of problem-solving, which is central to mathematics education. Although many students reported that they had not yet mastered this competence, they also indicated progress during practice. Nevertheless, this competence was rarely addressed explicitly in students' reflections – suggesting a need to guide students more systematically to reflect on problem-solving and related skills in order to strengthen their professional growth. This aligns with research showing that students who develop well-connected problem-solving schemas are better able to transfer their knowledge to new situations (Steele & Johanning, 2004; Hackenberg, 2010). In this light, our results send a clear message: mathematics teaching must prioritize the development of students' problem-solving schemas. To support this process, teachers should design and present problems that deliberately foster the growth of such schemas (Hodnik & Manfreda Kolar, 2015).

Overall, the findings reinforce that successful mathematics teaching requires more than mastery of subject knowledge. Students' reflections show the greatest emphasis on competences within the scope of PCK, highlighting their awareness

of the importance of didactical and pedagogical skills alongside content expertise. These results are consistent with Dewey's (1933) view of reflection as active and deliberate consideration of experience and Schön's (1983) emphasis on the importance of reflecting after action.

8. Conclusion

Teacher professional development is a lifelong process through which teachers expand their knowledge and refine their teaching methods (Valenčič Zuljan et al., 2012). While experience is important, teachers learn most effectively when they actively reflect on their practice (Lipton & Wellman, 2001). Reflection lies at the heart of professional growth, allowing teachers to give theoretical meaning to their actions and experiences, which in turn informs future practice. Combining experience with theoretical knowledge constitutes experiential learning, a model that helps future teachers optimally shape their practical experience during their studies (Marentič Požarnik, 2007).

Our research shows that future teachers reflect on the key knowledge and skills needed for quality teaching, recognizing that effective instruction requires more than mastery of subject matter alone (Ball et al., 2008; Carrillo et al., 2018). Teaching practice enables students to develop a wide range of competences, particularly those that are closely tied to classroom experience, such as performance and communication skills. However, our findings also reveal that certain areas, such as problem-solving skills, are often neglected in students' reflections, highlighting the need for more systematic guidance. According to Lester et al. (1989) there is a dynamic interaction between mathematical concepts and the metacognitive processes used to solve problems with these concepts. This means that awareness of cognitive processes develops in parallel with the understanding of mathematical concepts. Structured written reflection accompanied by appropriate feedback could foster deeper understanding of one's own practice (Cohen-Sayag & Fischl, 2012; Lane et al., 2014). In this context, students' practice diaries can be considered an example of guided reflection.

Overall, our findings indicate that future mathematics teachers develop an integrated combination of mathematical and didactic knowledge and beliefs that inform their teaching practice. Targeted development of didactic-mathematical knowledge and reflective skills is often more effective than merely strengthening subject knowledge (Zolfaghari, 2020). This aligns with theoretical models that integrate knowledge, beliefs, and reflection (Shulman, 1986; Ball et al., 2008; Carrillo et al., 2018; Korthagen, 2004, 2010) while also supporting the development of metacognitive and self-regulatory processes (Flavell, 1979; Schraw & Moshman, 1995; Pintrich, 2002; Schunk, 2008). Our research therefore contributes to understanding how pedagogical practice can effectively integrate theory, practice, and reflection, promoting sustainable professional development for teachers.

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Contact addresses:

Vida Manfreda Kolar
Faculty of Education, University of Ljubljana
Kardeljeva ploščad 16, 1000 Ljubljana, Slovenia
e-mail: vida.manfreda-kolar@pef.uni-lj.si

Adrijana Mastnak
Faculty of Education, University of Ljubljana
Kardeljeva ploščad 16, 1000 Ljubljana, Slovenia
e-mail: adrijana.mastnak@pef.uni-lj.si

Pogledi na samorefleksijo in samoocenjevanje razvoja učiteljskih kompetenc pri študentih matematike

Vida Manfreda Kolar in Adrijana Mastnak

Pedagoška fakulteta, Univerza v Ljubljani, Slovenija

Povzetek. Številni raziskovalci poudarjajo vse bolj kompleksno in zahtevno vlogo učiteljev v hitro razvijajoči se družbi ter izpostavljajo pomen priprave bodočih pedagogov na vseživljenjsko učenje. Zato je cilj izobraževanja učiteljev razvijati širok nabor kompetenc, ki vključujejo predmetnospecifično znanje, pedagoške spretnosti in metodološke pristope. Enako pomembno je spodbujanje ozaveščenosti o lastnih prepričanjih glede poučevanja, saj ta močno vplivajo na to, kako bodoči učitelji dojemajo in interpretirajo učne situacije ter izobraževalne dosežke. Za podporo temu razvoju je ključno, da bodoči učitelji pridobijo ustrezno strokovno znanje, ki ga lahko učinkovito uporabijo v pedagoški praksi, hkrati pa jih je treba spodbujati k stalnemu izboljševanju skozi samorefleksijo. Prispevek obravnava vlogo samoocene in refleksije študentov o lastnem znanju v okviru oblikovanja njihove učiteljske identitete.

V raziskavo so bili vključeni študenti dvopredmetnega študijskega programa matematike na Pedagoški fakulteti, ki so tekom študija pridobili potrebno strokovno in didaktično znanje za poučevanje matematike. Raziskava temelji na analizi dnevnikov pedagoške prakse, zbranih v dveh zaporednih študijskih letih (tretje in četrto leto). Osredotoča se na dve ključni področji: študentovo samoocenjevanje kompetenc za poučevanje matematike ter njihovo samorefleksijo izvedenih učnih ur. Glavni namen je prepoznati področja, kjer študenti zaznavajo vrzeli ali napredek v svojih kompetencah ter razumeti, katerim vidikom poučevanja namenjajo največ pozornosti pri refleksiji. Poleg tega raziskava proučuje morebitne razlike v samoocenjenih kompetencah med tretjim in četrtim letnikom študija. Z analizo dnevnikov prakse želimo pridobiti vpogled v to, kako študenti dojemajo sebe, svojo vlogo in svoje znanje kot bodoči učitelji matematike. Ugotovitve bodo prispevale k nadaljnjemu razvoju podpore in opolnomočenja študentov pri oblikovanju samozavestnih, kompetentnih in reflektivnih pedagogov.

Ključne besede: študent matematike, pedagoška praksa, refleksija poučevanja, matematične kompetence, samoocena kompetenc

Transitions in STEM Education – A Systematic Literature Review

Karolina Dobi Barišić* and Ana Katalenić

Faculty of Education, Josip Juraj Strossmayer University of Osijek, Croatia

Abstract. Society continues to show strong interest in STEM education, driven by various agendas. Support for STEM education significantly develops students' reasoning and problem-solving skills while increasing their likelihood of pursuing careers in related fields. Approaches to STEM education vary based on the type of education, whether formal or informal, the educational level ranging from pre-school, primary, secondary to university education, and curriculum focus, whether goal-oriented or content-driven. Given the growing interest in transitional issues within education, this study aims to review existing research on transitions in STEM education. The transitional issues pertain to students' adaptation to changes between different educational contexts, particularly between consecutive educational levels. This research employs the PRISMA methodology for systematic reviews, ensuring a scientifically grounded selection of relevant studies. This study analyses the methodological characteristics of the selected research, the perspectives on STEM education at specific educational levels, and the transitional issues identified in the literature. The findings provide valuable insights for educational practice, policymaking, and future research in STEM education.

Keywords: STEM, education, transition, PRISMA

1. Introduction

The term STEM (**S**cience, **T**echnology, **E**ngineering, and **M**athematics) refers to the aggregation of the broad fields of natural sciences, mathematics and statistics; information and communication technologies; and engineering, manufacturing and construction (OECD, 2024). The demand for STEM jobs is high, especially in industries that require digitized and creative approaches (Bose & Bäckman, 2020).

Many strategic documents at the national and European Union levels emphasize the importance of STEM education and the development of STEM competencies

*Corresponding author.

among pupils and teachers. The Strategy for Education, Science, and Technology is one of the Republic of Croatia's first strategic documents, emphasizing the importance of acquiring STEM skills (Vlada Republike Hrvatske, 2014). The National Education System Development Plan until 2027 (Ministarstvo znanosti i obrazovanja, 2023) aims to improve STEM skills of teachers and educational staff, as well as enhance students' STEM skills. National Recovery and Resilience Plan 2021–2026 (Vlada Republike Hrvatske, 2021b), under component 3.2. – Education, clearly establishes investments in STEM research infrastructure, as well as digital infrastructure, and needs to increase the number of students in STEM fields. The National Development Strategy of the Republic of Croatia 2030 (Vlada Republike Hrvatske, 2021a) emphasizes the importance of STEM education and research with the aim to empower Croatia's economic development, including the need to attract and retain quality STEM educators. The mentioned documents have the common goal of creating modern education, science, and research through the implementation of digital and green transitions. All this could be achieved through development of STEM education.

The Council of the European Union (2018), by listing competences for lifelong learning, emphasizes the importance of acquiring digital, mathematical, and scientific skills for all European Union citizens. In order to achieve a high level of competitiveness and technological leadership, EU STEM Education Strategic Plan (European Commission, 2025) emphasizes the need for a stronger focus on STEM. In the document, it is highlighted that there is a lack of STEM graduates and that there is a gender gap, which made the inclusion of more girls in STEM careers a necessity. The plan calls for a long-term STEM education strategy across the entire education system. The Digital Education Action Plan 2021–2027 (European Commission, 2023) supports initiatives to increase the number of women and girls in STEM, through financing training programs, and helping establish STEM platforms at national level to make engineering and ICT training programs more accessible.

2. Literature review

The previous studies on STEM education have mostly been conducted in the USA and generally Western countries (Li et al., 2020, 2022; Noris et al., 2023; Zhan et al., 2022). The USA has the largest concentration of research because it was a leader in reflection about modern STEM initiatives in education (Li et al., 2020). Since 2017, Eastern and developing countries have increasingly joined in research on STEM education (Zhan et al., 2022). Research in the STEM field mainly focused on learning and teaching, specifically on pedagogical practices, curriculum, and understanding concepts (Noris et al., 2023). Li et al. (2020) analysed journals from the field of STEM education, and identified factors of educational goals, educational policy, curriculum, assessment, and evaluation as important. Regarding the research methods used, there were differences according to the source of publications. An overview of research available in the Web of Science database showed that quantitative methods predominate, followed by qualitative methods,

and, finally, combined research methods (Li et al., 2022). In contrast, studies published by Taylor & Francis were mostly qualitative while quantitative and mixed methods were less common (Noris et al., 2023). Until 2000, studies in STEM education were rare, and after 2010, a slight increase was detected. The most important period was after 2015, when the number of publications significantly increased (Portillo-Blanco et al., 2024; Zhan et al., 2022). In 2015, the USA, the UK and Australia adopted national strategies for the development of STEM education, pointing it out as a priority field of development and research, which had a major impact on research in that area (Zhan et al., 2022). A literature review conducted by Portillo-Blanco et al. (2024) highlighted that most studies on STEM education were published between 2017 and 2023.

The focus of earlier research has most often been on individual STEM disciplines (e.g., mathematics), while contemporary research is multidisciplinary and interdisciplinary (Li et al., 2022; Zhan et al., 2022), ensuring a holistic approach to education, which ensures easier transfer of knowledge between disciplines, but also equips students to solve real-life problems (Al Hamad et al., 2024). STEM education is researched at different educational levels, including primary, secondary, and university, with each level having its own specificities, challenges and opportunities for effective STEM education (Di Martino et al., 2023). Professional development of teachers in STEM education has been recognized as a key factor in improving the quality of teaching and encouraging student engagement in STEM subjects (Yelland & Waghorn, 2023).

In the scientific literature, transitions between different educational levels and institution were mostly linked to the context of social, psychological and academic factors as the main prerequisites for a successful transition. The factors identified in the literature included different ecosystems such as family, peers, other students, school and community (Evans et al., 2018; Hosokawa & Katsura, 2019; Jindal-Snape et al., 2020; Kiuru et al., 2020; Tett et al., 2017). Some studies focused on specific situations in educational transitions, such as educational transition in the context of mathematics education (Montecino & Andrade-Molina, 2024), transition from face-to-face to online classes during the COVID-19 pandemic (Gin et al., 2021; Turnbull et al., 2021), or educational transitions of students with disabilities (Gin et al., 2021; Goodall et al., 2022). In the context of educational transitions, the literature considered transitions between kindergarten and primary school (Hosokawa & Katsura, 2019), primary and secondary school (Kiuru et al., 2020), secondary school and university (Akiha et al., 2018), university and postgraduate studies (Brathwaite, 2023), but also transitions from different levels of the education system to the labour market (Goodall et al., 2022), as well as transitions within different levels of higher education (Lau et al., 2022).

2.1. Research questions

One of the critical issues in STEM education is the transition between different educational levels, which can significantly impact students' continued interest and success in STEM fields (Murphy et al., 2020). Previous review studies rarely

systematically examined educational transitions in the context of STEM education. This paper aims to systematically review the literature on transitions between educational levels in STEM education to determine the methodological characteristics of such research, the transitions between institutions or educational levels that have been studied, and the topics represented in this research. For that reason, we pose the following research questions related to such studies:

RQ1: What are the methodological characteristics of the studies?

RQ2: Which educational or institutional transitions are addressed in the studies?

RQ3: Which educational topics are addressed in the studies?

3. Methodology

The study is a systematic review employing the PRISMA 2020 (Preferred Reporting Items for Systematic Reviews and Meta-Analyses) methodology to examine research related to STEM education and educational transitions (Page et al., 2021). PRISMA guidelines provide a transparent and reliable method for systematic analysis of a large body of research. The review included three steps: identification, screening, and inclusion for further analysis. We performed a data-driven qualitative content analysis of abstracts of included studies (Kuckartz, 2019).

Inclusion criteria for the identification step were (1) studies mentioning STEM, education, and transition, and (2) studies published after 2010. Identification process was performed on April 11, 2025 in Scopus and WoS databases. The search string with Boolean operators "STEM AND education AND transition" was applied on "Title, abstract, keywords" in Scopus and on "Topic (Title, Abstract, Author Keywords, Keywords plus)" in WoS. A filter "From 2010" was applied on search results in both databases. The records were exported to excel file and duplicates across two databases were removed in this step. Each entry contained the following meta-data: authors, study title, year, source title, keywords, abstract.

The screening process included two steps, screening for suitable records and eligible studies. The suitability was assessed on three counts: (1) the record referred to a single research study, (2) it included a complete abstract, and (3) it was in English. The eligibility was accounted based on the following exclusion criteria:

(1) The study was not focused on education

(2) The study was not focused on STEM education

(3) The study focused on STEM education outside transition context

(4) The study focused on education transition outside STEM context

The data-driven content analysis comprised four steps, wherein the initial step of preparing the data was achieved via PRISMA systematic review. In the next step, we established the main classes of categories related to our research questions: (1) STEM area addressed in the study, (2) methodological characteristics of the studies related to (a) study location, (b) study design, (c) study sample, (3) transition

considered in the study, and (4) research topic of the study. The step corresponding to the coding of the data was an inductive process revolving around interpreting data, assigning theme, and cross-referencing entries until saturation occurs. Each of the four classes would thus comprise related analytic categories distinguishing between studies with similar characteristics. The final step was category-based analyses and presenting results using descriptive statistics.

The screening process was divided between the authors in such a way that each author reviewed a portion of the retrieved records according to predetermined eligibility criteria.

4. Results

The PRISMA 2020 flow diagram, shown in Figure 1, represents the study selection process across the three steps: identification, screening, and inclusion. The implemented search string resulted with a total of 1463 records in two databases; 406 records were initially removed from the cumulative database, either as duplicate records or published before year 2010. The first screening found a total of 32 unsuitable

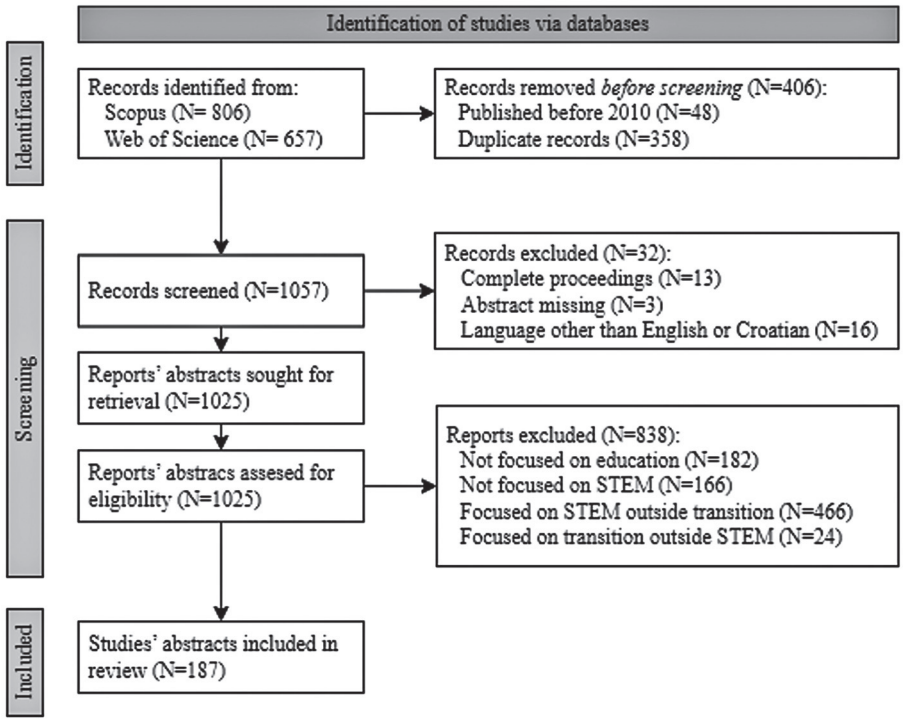

Figure 1. PRISMA 2020 flow diagram for a "STEM education transition" systematic review.

records based on the predetermined criteria. Second screening assessed a total of 1025 records for eligibility. Majority of excluded studies (N = 466) examined some aspect of STEM education unrelated to educational or institutional transition, e.g., gender research or design studies at single educational level. Studies not focused on STEM or education (N = 348) contained words "stem", "education", and "transition" with different meanings than those relevant to the study in the searched fields. Ultimately, abstracts of 187 scientific papers were included in this review.

The number of studies related to "STEM education transition" increased over 2010–2024 (Figure 2). Up until the retrieval date, seven studies were published in 2025.

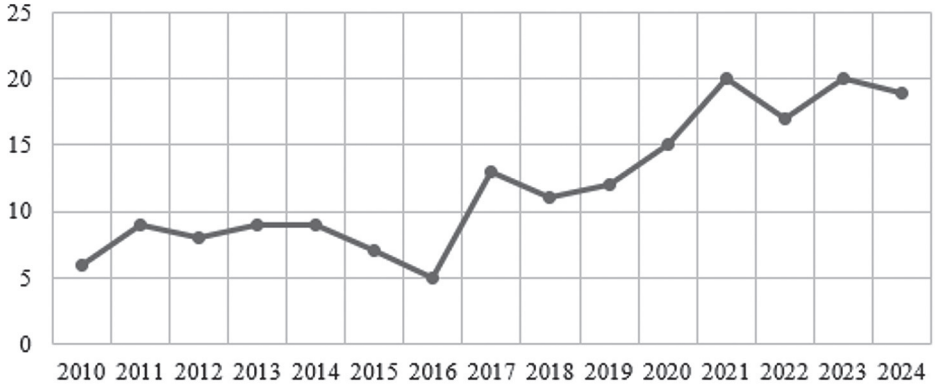

Figure 2. Trends in the number of studies from 2010 to 2024.

The categories related to STEM fields included corresponding fields or combination of fields. The majority of studies concerned STEM education in general, and many studies focused on the role of Mathematics or Engineering in STEM

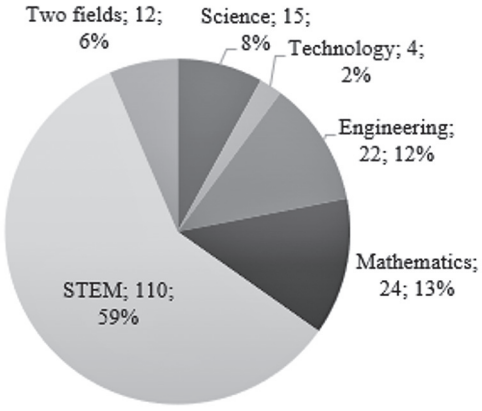

Figure 3. Distribution of studies in terms of the STEM field focus.

education. Studies focused on Science and Technology were fewer and more recent, from 2018 onward. The most common examined combination of fields was Engineering and Technology (N = 8). Three studies focused on the combination of Mathematics and Science, and one on the combination of Mathematics and Engineering.

4.1. Methodological characteristics of the studies

The categories related to the methodological characteristics of the study (location, design, and sample) and distribution of studies according to those categories are in Table 1. In each class, a significant number of records provided study abstract with insufficient information to determine the corresponding category. Only six studies ($\approx 3\%$) were coded with unknown category related to all methodological characteristics.

Table 1. Distribution of studies regarding the methodological characteristics.

	Category	**Count**	**Percentage**
Study location	United States of America	74	39.57%
	Unknown location	58	31.02%
	Other European countries	14	7.49%
	International (systematic reviews)	8	4.28%
	Australia	7	3.74%
	United Kingdom	7	3.74%
	Germany	5	2.67%
	Multiple European countries	5	2.67%
	Other non-European countries	4	2.14%
	Italy	3	1.60%
	Russia	2	1.07%
Study design	Qualitative study	45	24.06%
	Unknown design	36	19.25%
	Survey	30	16.04%
	Quantitative study	27	14.44%
	Design study	19	10.16%
	Mixed-methods study	12	6.42%
	Longitudinal study	9	4.81%
	Systematic review	9	4.81%
Study sample	University students	66	35.29%
	Unknown sample	34	18.18%
	Secondary school students	23	12.30%
	Alumni (graduate students)	17	9.09%
	Publications	17	9.09%
	Other samples	15	8.02%
	Complex samples	15	8.02%

Regarding the study location, the largest share of studies was situated in the USA. Only few other countries contributed with more than one study (Australia, the

UK, Germany, Italy, and Russia). One study was located in each of the countries: Austria, Belgium, Denmark, Finland, France, Hungary, Ireland, Norway, Portugal, Romania, Serbia, Slovakia, Spain, and Turkey from Europe, and Cambodia, Canada, Ethiopia, and Kazakhstan outside Europe. Few studies were set in multiple European countries: Norway, Sweden, and the UK; Finland, Spain, and the UK; Czech, Spain, and Portugal; Germany, Italy, and the UK; and Belgium, Finland, Germany, Hungary, Slovakia, and the UK. No studies related to educational transition in STEM was performed in Croatia. Some of the studies were systematic reviews covering the context of international research. The number of the USA studies remained approximately the same, while the number of studies from other countries increased over 2010–2025.

The categories related to study design included qualitative, quantitative, and mixed methods studies, surveys, design and longitudinal studies, and systematic reviews. The majority of studies were qualitative studies. Among the nine review studies, one considered only studies set in Russian context. Longitudinal and mixed-methods studies decreased, while qualitative, quantitative, and design studies, surveys, and systematic reviews increased over 2010–2025.

The categories related to study sample included primary and secondary school students, university and PhD students, parents, teachers, alumni, and publications, such as curriculum and research. Some studies had complex samples, including combinations such as, students on different educational levels, students and teachers, and curriculum and students. The majority of studies included university students. The number of studies including secondary school students remained approximately the same over 2010–2025, while the number of studies including university students, alumni, and publications increased. The latter is in line with the increase of systematic reviews.

4.2. Transitions considered in the studies

The categories related to the transition included transitions between multiple educational levels, between early childhood, primary school, and secondary school, from secondary school to labour market or university, within university, from university to labour market or PhD studies, from PhD to labour market, from labour market to university. The transitions to, from, and within university were most frequently examined in the studies. Majority of studies examined the transition from secondary school to university, and they accounted for nearly half of the studies over 2010–2025. The share of studies concerning the transition to and within university remained stable over the same period, while those concerning transition to labour market increased.

When comparing the study location to transition categories, the transition to and from university was covered across different contexts. The USA context covered the majority of studies concerning the transition within university ($\approx 72\%$), and no studies concerning the primary school. The Australian context covered the largest share of studies concerning the transition to or from primary school (30%).

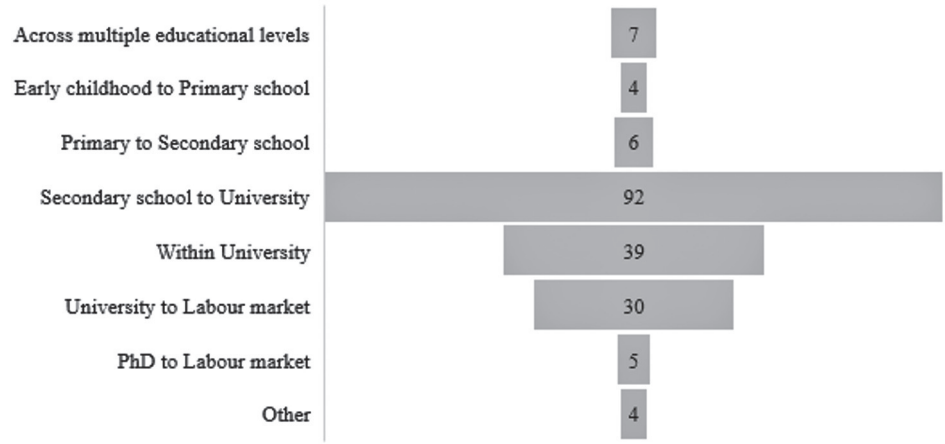

Figure 4. Distribution of studies in terms of the transition focus.

4.3. Topics covered in the studies

The topics covered in the studies were grouped around nine categories: student or teacher perspective, pedagogy, curriculum, didactics, or employment focus, course/programme or government strategy evaluation, and other specific issues. The distribution of studies according to these categories is in Figure 5. Nearly half of the studies focused on course or programme evaluation, and their number was approximately the same over 2010–2025. Studies focused on students, curriculum, employment, and specific topics increased in the same period.

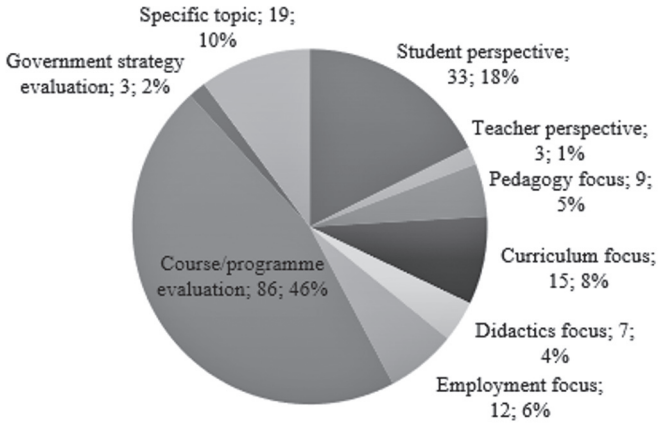

Figure 5. Distribution of studies in terms of the topic covered.

Studies focused on curriculum (40%) and employment ($\approx$ 42%) were common in Europe, and majority of course/programme evaluation studies were set in the USA ($\approx$ 65%). The student perspective was mainly considered in studies

on secondary-to-university transitions ($\approx 61\%$). Course/programme evaluation ($\approx 95\%$) and specific issues ($\approx 89\%$) were mainly addressed in studies covering transition to, within, and from university. More specifically, over half of the studies on course/programme evaluation considered secondary-to-university transition ($\approx 55\%$).

5. Discussion and conclusion

The paper included a systematic review and a content analysis of studies related to educational transitions in STEM. Results showed that the majority of studies were situated in the USA, which is in line with the research of Li et al. (2020, 2022), Noris et al. (2023), and Zhan et al. (2022). Studies engaged university students more often than other groups and implemented a variety of study designs. Research related to STEM education includes different educational levels (Di Martino et al., 2023), and the most common direction of research is educational policy and curriculum (Li et al., 2020; Noris et al., 2023). Studies in this research mainly focused on the transition from secondary school to university and course/programme evaluation.

Search string creation was based solely on the research objective. Expanding the search string with additional keywords could have ensured retrieval of a wider corpus of relevant literature. In future research the search string should be expanded with larger scope of keywords. Considering that all retrieved records were not analysed by both reviewers, although pre-defined criteria were applied, deviations in the study selection are possible. This research, which is based on study selection abstracts, showed the geographic dominance of research conducted in the USA, but the selection of search databases, the language of the papers (English or Croatian) and the search strategy, i.e. search string, should be taken into account.

Our systematic review raised several questions related to pedagogy, curriculum, and future of universities: How to align and overcome differences across institutions? How to align and integrate content across subject areas? What is the future and the role of enrolment, bridging, or capstone courses and programs in university education worldwide, and in Croatia? Considering the results of our studies, the first two questions were not adequately addressed in previous studies, and the latter question was extensively addressed in the USA context.

Institutions at different educational levels in Croatia, from early childhood to university, adopt significantly different pedagogical approaches. Given the limited number of studies addressing this issue, we question whether other countries have more aligned pedagogical structures across educational levels or if there is a general lack of research on this topic. Further research should answer the question: How do these varied pedagogical perspectives across educational levels influence STEM education?

In Croatia, STEM areas are treated as distinct subjects as early as the primary level. Although STEM education is associated with an integrative approach and problem solving, the presence of these methods in curricula varies depending on

the educational level. Given the scarcity of research, future studies should include following questions: How have different educational contexts addressed the integration and problem-solving requirements of STEM education? How are these elements reflected in the intended and implemented curriculum in Croatia?

Considering the prevailing research focus on transitions in STEM education, we ask whether universities in Croatia and other countries – particularly outside the USA – respond similarly to the increasing demand for supporting students in their transition from secondary education to university and ultimately into the profession. How do these demands shape the ecology of university education and the broader community? What subject-matter and pedagogical knowledge will be required of university teachers to meet these challenges? What kinds of courses should we offer in the future, how should we organize our teaching, and how can we effectively engage students to grow within STEM fields?

The increasing trend in research on educational transition in STEM might provide answers to some of the questions arisen from this systematic review.

Data set

The data set, *Transitions in STEM education – systematic literature review – DATA SET*, used in this article, is created by the authors. It was developed to identify research focused on transitions between different educational levels in the context of STEM education. The research review method is explained in the Methodology section of this paper. After inclusion/exclusion phase, 187 abstracts were analysed and included in this review. All research results identified in Scopus and Web of Science based on search string "STEM AND education AND transition" are included in this database with the following information: author, study title, study year, source title, abstract, author keywords, index keywords, doi, inclusion/exclusion criteria. The data set is openly available for use by other researchers, with proper citation, at `https://urn.nsk.hr/urn:nbn:hr:141:079096`.

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Contact addresses:

Karolina Dobi Barišić
Faculty of Education, Josip Juraj Strossmayer University of Osijek
Cara Hadrijana 10, 31 000 Osijek, Croatia
e-mail: kdobi@foozos.hr

Ana Katalenić
Faculty of Education, Josip Juraj Strossmayer University of Osijek
Cara Hadrijana 10, 31 000 Osijek, Croatia
e-mail: akatalenic@foozos.hr

Prijelazi u STEM obrazovanju – sustavni pregled literature

Karolina Dobi Barišić i Ana Katalenić

Fakultet za odgojne i obrazovne znanosti, Sveučilište Josipa Jurja Strossmayera u Osijeku, Hrvatska

Sažetak. Potaknuto različitim razvojnim, obrazovnim i gospodarskim ciljevima, STEM obrazovanje kontinuirano izaziva veliki interes društva. Poticanje STEM obrazovanja doprinosi razvoju vještina zaključivanja i rješavanja problema kod učenika te se povećava mogućnost odabira karijere u STEM područjima. Pristupi STEM obrazovanju razlikuju se ovisno o vrsti obrazovanja, bilo da je riječ o formalnom ili neformalnom obrazovanju, obrazovnoj razini – od predškolskog, osnovnog, srednjeg do visokoškolskog obrazovanja – te ovisno o tome je li kurikulum usmjeren prema odgojno-obrazovnim ishodima ili sadržaju. S obzirom na rastući interes za izazove prijelaza u obrazovanju, cilj ovog rada je pregled postojećih istraživanja o prijelazima u STEM obrazovanju. Izazovi prijelaza odnose se na prilagodbu učenika promjenama između različitih obrazovnih konteksta, posebno između uzastopnih obrazovnih razina. Ovo istraživanje koristi PRISMA metodologiju sustavnog pregleda literature, čime je osigurana znanstveno utemeljena selekcija relevantnih istraživanja. U radu se analiziraju metodološke značajke odabranih istraživanja, perspektive STEM obrazovanja na određenim obrazovnim razinama te izazovi prijelaza identificirani u literaturi. Dobiveni rezultati pružaju vrijedne uvide za obrazovnu praksu, oblikovanje politika i buduća istraživanja u području STEM obrazovanja.

Ključne riječi: STEM, obrazovanje, prijelaz, PRISMA

The "Quantum Dots" (Chemistry Nobel Prize 2023) as an "Application" of My Non-Orientable Space Groups (Published in 1983–84) With Mistakes Improved

Emil Molnár

Department of Algebra and Geometry, Budapest University of Technology and Economics, Hungary

Submitted on 15 November 2025, on the occasion of the 80th *birthday of my late friend*

Miklós Freller

whose student circle I visited every year to hold a math lecture in Illyés Gyula Gymnasium, Dombóvár (Hungary), through more than 35 years. This paper is dedicated ***to his memory and to his kind family****, in fond remembrance of the happy occasions we spent together.*

Abstract. Reading a newspaper article from December 2023, I recognised that my 1984 conference paper may offer a possible explanation related to the Nobel Prize in Chemistry awarded to the laureates Alexey Yekimov, Louis E. Brus, and Moungi G. Bawendi. When I emailed my colleague, the chemist and academician István Hargittai, he congratulated me. I presented this topic at some conferences (Budapest, Sopron, Belgrade, Senj in 2024).

Of course, the present author could not have foreseen the relevance and importance of the topic at that time. That was an incidental consequence of my erroneous paper (1983) intended to construct an infinite series of non-orientable compact hyperbolic manifolds (space forms) as tricky polyhedral tiling series in the Bolyai-Lobachevsky hyperbolic space $\mathbf{H}^3$. Fortunately, I observed and improved the mistake soon. In fact, those constructions were not manifolds because of the two fixed point orbits as "punctures", where point reflections occurred in the symmetry group of the polyhedral tiling. But these singular points, as "quantum dots" e.g. for copper and chlorine ions, respectively, in glass (silicon) fluid, cause light effects (by "electrons jumping-leaping") whose colours might depend on the sizes of crystal particles. Think of the TV-screen (and other fluid crystals).

This means that the mistake was much more interesting than the original intention that could be reached "easily" soon!

Keywords: quantum dots, manifold and twice punctured (almost) manifold, Euclidean and hyperbolic crystal group, polyhedral tiling

1. Introduction. The Euclidean space group 61. Pbca, with brick fundamental domain, to twice punctured (almost) manifold $\mathbf{E}^3$/Pbca

We know the 17 plane crystallographic groups with compact fundamental domains (Hahn, 2005), as well as the 230 3-dimensional space groups.

Imagine a rectangle $OAEB$ in the Euclidean plane $\mathbf{E}^2$, also with a coordinate system: O as origin, $OA = \mathbf{e}_1$, $OB = \mathbf{e}_2$ as basis vectors, $OX = \boldsymbol{X} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 =: x^i\mathbf{e}_i$ (Einstein-Schouten summing index convention) as position vector of an arbitrary point $X(x^i)$ with coordinates (x^1, x^2). Consider the plane translations

$$\mathbf{t}_1 : (x^1, x^2) \to (x^1 + 1, x^2), \text{ or simply } x^1 + 1, x^2, \text{ mapping } OB \to AE;$$

$$\text{and analogously } \mathbf{t}_2 : \; x^1, x^2 + 1, OA \to BE.$$

These translations generate the plane group

1. p1 (the first plane group, as plane lattice): x^1, x^2 simply
now with specific orthorhombic lattice, given by pairs of integer coordinates, added to any above (x^1, x^2) with the identity, as trivial stabilizer. But the above two translations as gluing operations make *torus* ($\mathbf{E}^2/\mathbf{p}1$ as orbit plane) from the rectangle $OAEB$ (*fundamental domain*) as a *2-dimensional manifold, where any point (orbit) has a circle-disc-like neighbourhood (as the definition of manifold, now with Euclidean metric).* The vertices O, A, E, B *will be equivalent under group* **p1** (they are in the same orbit), their four vertex domains amount to a circular disc, indeed.

Now for another plane group **8. p2gg** (make a plane figure, please), we assume the above rectangle fundamental domain $OAEB$ to another coordinate system with O *as origin* but $2OA = \mathbf{e}_1$ and $2OB = \mathbf{e}_2$ as basis vectors and the analogous $\mathbf{t}_1, \mathbf{t}_2$ translation generators for the (larger) orthorhombic (primitive) lattice Λ_p. We find the typical transformations of **p2gg**, generated by $\mathbf{g}_2 = \mathbf{b} : OB \to EA$ and $\mathbf{g}_1 = \mathbf{a} : OA \to EB$ in Hahn (2005) in brief notations:

8. p2gg: (1) x^1, x^2 translation.
(2) $-x^1, -x^2$ point reflection: in the first orbit of $O(0,0)$ and $E(\frac{1}{2}, \frac{1}{2})$, or in the second orbit of $A(\frac{1}{2}, 0)$ and $B(0, \frac{1}{2})$ each is of half-circle-disc neighbourhood, i.e. ***these are the two punctures***.
(3) $-x^1 + \frac{1}{2}, x^2 + \frac{1}{2}$ glide reflection $\mathbf{g}_2 = \mathbf{b}$ with reflection line $\frac{1}{4}, x^2$.
(4) $x^1 + \frac{1}{2}, -x^2 + \frac{1}{2}$ glide reflection $\mathbf{g}_1 = \mathbf{a}$ with line $x^1, \frac{1}{4}$.
Glide reflections are fixed-point-free orientation reversing transformations, so orbits of type (3) and (4) each has a trivial stabilizer and hence a circle-disc-like neighbourhood. This means that the orbit space (plane) $\mathbf{E}^2/\mathbf{p}2\mathbf{gg}$ *is twice punctured (almost) plane manifold*.

But there is an essential (really hard) conceptual difference between plane orientation (either clockwise, i.e. negative or anti-clockwise, i.e. positive); and

space orientation (either positive (by the right-hand rule): if looking from the 3rd coordinate axis (the middle finger of the right hand) at the plane of 1st and 2nd axes, the 1st axis turns to the 2nd axis under positive (anti-clockwise) rotation; or negative (by the left-hand rule): in the opposite case).

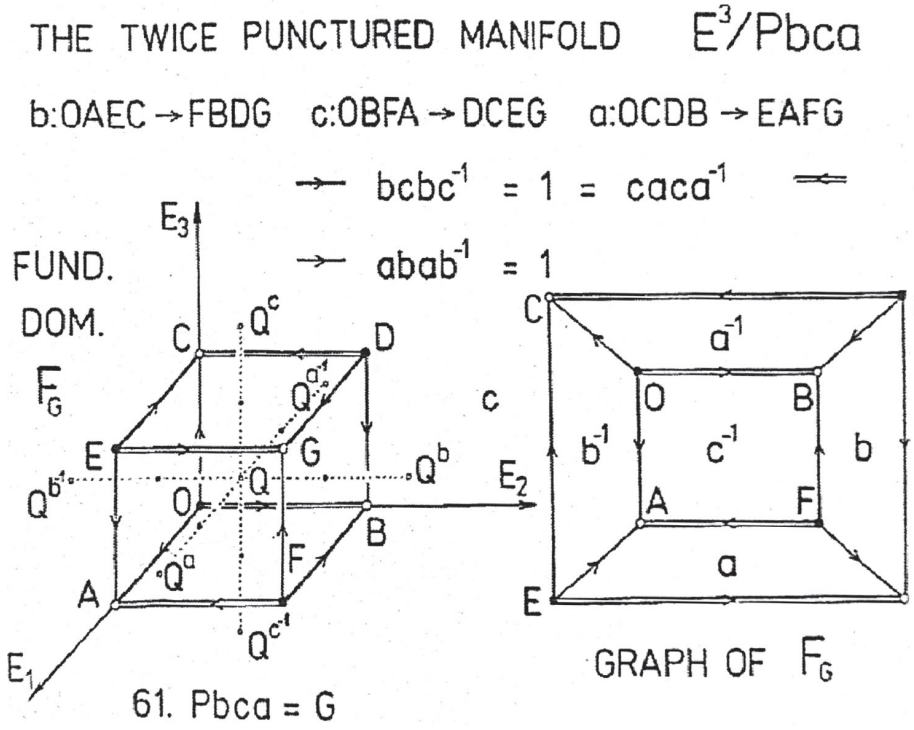


Figure 1. The Euclidean twice punctured (almost) manifold $\mathbf{E^3/Pbca}$.
Super group can occur: 205. $\mathbf{Pa\overline{3}} =: \mathbf{G}$, if $\mathrm{F_G}$ is a cube.
Then a 3-rotation is possible around the axis OG to a smaller fundamental domain, e.g. the pyramid $GOAFB$!

After this simpler crystallographic "gymnastics" we are ready to study Figure 1 with the crystallographic space group $\mathbf{G} = \mathbf{61.\ Pbca}$, our main phenomena with obvious (but not easy changes, more briefly). At Fig. 1 we look the brick fundamental domain $\mathrm{F}_G = OAFBCEGD$, analogously as before, but now with glide reflections **b**, **c**, **a** pairing the opposite faces in Fig. 1 as generators of **G**. These imply the following:

61. Pbca. *Orientation preserving transforms*:

(1) x^1, x^2, x^3 identity and translations generated by $\mathbf{t}_1$: $2OA = \mathbf{e}_1$, $\mathbf{t}_2$: $2OB = \mathbf{e}_2$, $\mathbf{t}_3$: $2OC = \mathbf{e}_3$ for orthorhombic (primitive) lattice Λ_P.

(2) Screw motion with half-turn $\mathbf{s}_1$: $\frac{1}{2} + x^1, \frac{1}{2} - x^2, -x^3$ with screw line $x^1, \frac{1}{4}, 0$.

(3) Screw motion $\mathbf{s}_2$: $-x^1, \frac{1}{2} + x^2, \frac{1}{2} - x^3$ with screw line $0, x^2, \frac{1}{4}$.

(4) Screw motion $\mathbf{s}_3$: $\frac{1}{2} - x^1, -x^2, \frac{1}{2} + x^3$ with screw line $\frac{1}{4}, 0, x^3$. Screw motions are fixed-point-free transforms, so any orbit has trivial stabilizer and ball-like neighbourhood.

Orientation reversing transforms:

(5) $-x^1, -x^2, -x^3$ point reflections (in the origin O), so by the lattice Λ_P in two orbits: of O, D, E, F and of G, A, B, C. ***These serve the two punctures***, each with half-ball-like neighbourhood, where opposite points of a ball are “glued together”. If we glue together the four corner domains, we realize this in fact.

(6) Glide reflection **b**: $\frac{1}{2} - x^1, \frac{1}{2} + x^2, x^3$: $OAEC =: b^{-1} \to FBDG =: b$ with glide plane $\frac{1}{4}, x^2, x^3$.

(7) Glide reflection **c**: $x^1, \frac{1}{2} - x^2, \frac{1}{2} + x^3$ with glide plane $x^1, \frac{1}{4}, x^3$.

(8) Glide reflection **a**: $\frac{1}{2} + x^1, x^2, \frac{1}{2} - x^3$ with glide plane $x^1, x^2, \frac{1}{4}$. Glide reflections are fixed-point-free transforms, so any orbit has ball-like neighbourhood.

At later presentations a movable “wonder cube” model (recently presented to me by Endre BUDAI, a teacher colleague of Teleki Blanka Gymnasium in Székesfehérvár (Hungary), in the name of Gymnasium, on the occasion of my Dezső Lázár Memorial Student Circle Lecture 2024 April, found the Internet link https://www.thingiverse.com/thing:10483) just incidentally animating my Fig. 1. The opposite vertices with bigger part (Cu), smaller part (Cl) can be threefold rotated, like another Rubik’s cube.



Figure 2. A “Wonder Cube” model (by Endre Budai, teacher of Teleki Blanka Gymnasium, Székesfehérvár).

Péter SZABÓ, University of Sopron, produced this cube model very smoothly by a 3D printer in more variants.

The above fundamental domain (asymmetric unit in Fig. 1) $OAFBCEGD = \mathrm{F}_{\mathbf{G}}$ of **61. Pbca** $= \mathbf{G}$ and the face pairing generated space tiling $(\mathrm{F}_{\mathbf{G}})^{\mathbf{G}}$ geometrically describes this group $\mathbf{G}$ (e.g., Hahn, 2005, pp. 296–297) and the crystallographic tiling (by material particles) in the orthorhombic coordinate system $OE_1E_2E_3$. The lengths of basis vectors $|OE_i| := |\mathbf{e}_i| = a_i$ $(i = 1, 2, 3)$ are given parameters (by measuring the material crystal in nano size, to be determined, 1 nanometre $= 10^{-9}$ meter). An orthorhombic lattice Λ_G of $\mathbf{G}$ is given by integer coordinate triples (only) to the identity transform $\mathbf{1}$ (as linear part). As we know (from linear algebra in our conventions), each transform $\boldsymbol{\alpha}(\mathbf{A}, \mathbf{a}) \in \mathbf{G}$ can be given by mapping $\boldsymbol{\alpha} : \boldsymbol{X} \to \boldsymbol{X}\mathbf{A} + \mathbf{a} =: \boldsymbol{X}^{\boldsymbol{\alpha}} =: \boldsymbol{Y}$ with $\mathbf{A}$ by linear integer unimodular matrix to the basis $(\mathbf{e}_i = OE_i)$ above, and with the broken part $\mathbf{a}$, the position vector is $OX = \boldsymbol{X} = x^i\mathbf{e}_i$ (Einstein-Schouten summing index convention), the above $\boldsymbol{Y}$ is the image of $\boldsymbol{X}$ at $\boldsymbol{\alpha}$.

We can see that O, D, E, F are $\mathbf{G}$-equivalent point reflection centres, say with copper (Cu) ion parts, so are G, A, B, C with chlorine (Cl) ion parts, so that the fundamental cube $\mathrm{F}_{\mathbf{G}}$ can (assume) contain also "central" silicon (Si) atom in OG and 2-2 ones at the 3 opposite face pairs of $\mathrm{F}_{\mathbf{G}}$, equivalent by glide reflections $\mathbf{b}, \mathbf{c}, \mathbf{a}$, respectively. Imagine that these $\frac{1}{2} : \frac{1}{2} : 7(?)$ proportions of atoms Cu, Cl, Si can form crystal particles with appropriate cube size, and these particles (with more cubes) float in a silicon fluid that freezes later. The singularities (as constraints) near Cu and Cl ions cause electrons "jumping-leaping" with light effects, i.e. quantum dots. The colour of light can depend on the size of crystal particle. Imagine the television screen! The Periodic Table of Elements suggests other material possibilities as well.

2. Hyperbolic series

We can imagine that in nano size (again, 1 nm $= 10^{-9}$ m) hyperbolic (or spherical and other) geometry can also occur (in certain physical circumstances). So please, consider the crystallographic world with János Bolyai's "Absolute Geometry", now in hyperbolic or Bolyai-Lobachevsky geometry. In $\mathbf{H}^2$ plane any triangle has the angle sum less than $180^\circ = \pi$. In $\mathbf{H}^3$ a deeper fact holds: any projective coordinate tetrahedron $b^0b^1b^2b^3 \sim A_0A_1A_2A_3$ (where $A_iA_jA_k = b_l$ for $\{i, j, k, l\} = \{0, 1, 2, 3\}$, as usual for vertices A_i and faces b^j of a tetrahedron), we assume that the Coxeter-Schläfli angle matrix ((C-Sch) briefly, or cosine matrix):

$$(-\cos(b^ib^j)) = (\langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle) =: (b^{ij}) \tag{2.1}$$

is of hyperbolic signature $(+++-)$ for the "sign changes of main minor determinants" (see later). Here (b^ib^j) denotes the angle of faces b^i, b^j; by convention let $(b^ib^i) = 180^\circ = \pi$. In the following we consider only doubly truncated orthoscheme of (C-Sch) diagram:

$$0\!-\!\!-\!6\!-\!\!-\!1\!-\!6\!-\!2\!-\!6\!-\!\!-\!3. \tag{2.2}$$

That means $(b^0b^1) = (b^1b^2) = (b^2b^3) = \pi/6$ but the other three angles are $\pi/2$ right angles. We can check the above criterion and see that the angle sum of

side planes at A_0 and A_3 are $2\pi/6 + \pi/2 < \pi$. So A_0 and A_3 are outer vertices (of the Beltrami-Cayley-Klein (B-C-K) projective model of $\mathbf{H}^3$). As it is "well known" from the elementary projective hyperbolic geometry, the corresponding polar planes a_0 and a_3 orthogonally intersect the side planes, except b^0 and b^3, respectively. The distance $b^0a_0 = b^3a_3$ and the other distances of the orthoscheme can also be computed (later in Fig. 3) by the projective metric as well, e.g. Molnár and Szirmai (2024) and previous works of the author. Monographs of Vinberg (1993) and Wolf (1967) are the basic references.

In the general linear algebraic theory, $\mathbf{H}^3$ is defined over a real projective metric vector space $\mathbf{V}^4$ with basis $\{\mathbf{a}_i\}$, their dual form space $\boldsymbol{V}_4$ with dual basis $\{\boldsymbol{b}^j\}$ just to the above coordinate tetrahedron (simplex) $b^0b^1b^2b^3 \sim A_0A_1A_2A_3$ with projective description $A_i(\mathbf{a}_i), b^j(\boldsymbol{b}^j)$ with proportional equivalence $\sim$: $\mathbf{a}_i \sim c\mathbf{a}_i$ for point A_i and any point $X(\mathbf{x} = x^i\mathbf{a}i \sim cx), \boldsymbol{b}^j \sim \boldsymbol{b}^jc$ for plane b^j and any plane $u(\boldsymbol{u} = \boldsymbol{b}^ju_j \sim \boldsymbol{u}c)$; and c is any real number except 0 (zero). We assume $\mathbf{a}_i\boldsymbol{b}^j = \delta_i^j$ the Kronecker symbol for the above coordinate simplex. For hyperbolic geometry we define a linear symmetric (polar) plane $\rightarrow$ (pole) point, $u(\boldsymbol{u}) \rightarrow U(\mathbf{u} = u\blacktriangledown)$ polarity mapping from $\mathbf{V}_4$ into $\mathbf{V}^4$ by $\boldsymbol{b}^j\blacktriangledown = \mathbf{b}^j = b^{jk}\mathbf{a}_k$ with symmetric matrix $b^{jk} = b^{kj}$ or scalar product

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle := \boldsymbol{u} \blacktriangledown \boldsymbol{v} = (\boldsymbol{b}^j\blacktriangledown u_j)(\boldsymbol{b}^k v_k) = u_j b^{js}\mathbf{a}_s\boldsymbol{b}^k v_k = u_j b^{js}\delta_s^k v_k = u_j b^{jk} v_k. \tag{2.3}$$

We can see (not easily) that (2.1) is in accordance with (2.3). Thus in general, the label 6 in (2.2) is replaced by $2q$ ($q = 3, 5, 7, \ldots$ odd numbers $q = 2t + 1$ later, e.g. in Fig. 3), and we can compute the main minor determinants of (2.1) with angle $\pi/2q =: \zeta$ and check the signature $(+++-)$, by main minor determinants to (2.1) and (2.2); or by Gram-Schmidt orthogonalisation (diagonalisation) as follows:

$$1;\ \sin^2\zeta \in \left(0, \frac{1}{4}\right];\ 1 - 2\cos^2\zeta = -\cos(2\zeta) \in \left(-1, -\frac{1}{2}\right];$$

$$\begin{aligned} B :&= \det(\langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle) = \sin^4\zeta - \cos^2\zeta \\ &= (\sin^2\zeta - \cos\zeta)(\sin^2\zeta + \cos\zeta) \in \left(-1, -\frac{11}{16}\right] \end{aligned} \tag{2.4}$$

with sign changes $+ \rightarrow + \rightarrow - \rightarrow -$, i.e. with signature $(++-+) = (+++-)$ indeed, also in general by (2.1), (2.4).

By (2.3) we can compute the angle (uv) of planes $u(\boldsymbol{u})$ and $v(\boldsymbol{v})$:

$$\cos(uv) = -\frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\sqrt{\langle \boldsymbol{u}, \boldsymbol{u} \rangle \langle \boldsymbol{v}, \boldsymbol{v} \rangle}} \tag{2.5}$$

also by complex extension through complex exponential. We can compute the inverse matrix of (b^{ij}), denoted by $(b^{ij})^{-1} = (a_{ij})$, according to inverse polarity, by (2.1) and (2.3), and scalar product $\langle\,,\rangle$ in $\mathbf{V}^4$:

$$(a_{ij}) = (\langle \mathbf{a}_i, \mathbf{a}_j \rangle) = (b^{ij})^{-1}. \tag{2.6}$$

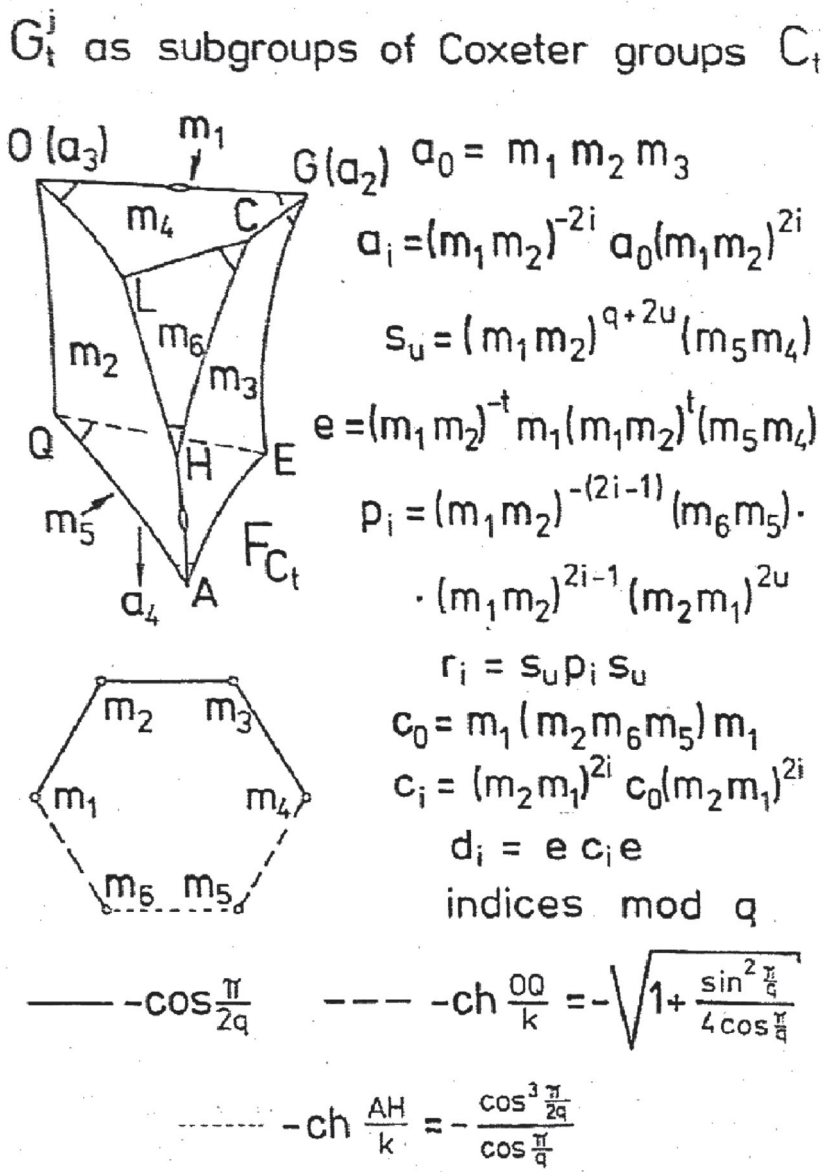


Figure 3. From the original paper Molnár (1987), the doubly truncated orthoscheme comes from the $A_0A_1A_2A_3 = b^0b^1b^2b^3$ projective coordinate simplex, but here we used the previous notations: $\boldsymbol{A}_2 \to \boldsymbol{a}_3$ for point O, $\boldsymbol{A}_1 \to \boldsymbol{a}_2$ for point G, $\boldsymbol{A}_3 \to \boldsymbol{a}_4$ for outer point A_3, whose polar plane a_3 is m_5 here, A_0 is outer point whose polar plane a_0 is denoted by m_6 here. The simplex planes b^0, b^1, b^2, b^3 are denoted by m_1, m_2, m_3, m_4 now (as mirror planes for Coxeter reflection group $\mathbf{C}_t$). The computations will be more clear later on. Our following disc-like fundamental domains will be glued together from this doubly truncated orthoscheme (hopefully in an obvious manner, by plane reflections).

So the distance $d(\mathbf{x},\mathbf{y}) =: |XY|$ of points $X(\mathbf{x})$ and $Y(\mathbf{y})$ (and also its complex extension) can be computed as well:

$$\cosh\left[\frac{d(\mathbf{x},\mathbf{y})}{k}\right] = -\frac{\langle \mathbf{x},\mathbf{y}\rangle}{\sqrt{\langle x,x\rangle\,\langle \mathbf{y},\mathbf{y}\rangle}} \text{ with } k = \sqrt{-\frac{1}{K}}, \tag{2.7}$$

K is the negative sectional curvature, so k is the natural length unit (to be measured in praxis).

2.1. First fundamental domain series of groups resulting in twice punctured (almost) manifolds in hyperbolic space H^3

To our first series for simplicity, we shall discuss only the starting case in Fig. 4 to the hexagonal disc-like fundamental domain, i.e. $q = 3, t = 1, u = 0$ the most symmetric case ($q = 5$, i.e. decagonal fundamental domain has been treated in our previous papers with more details; of course, we need your fantasy to all infinitely many cases).

We indicate our infinitely many cases first for domain and group by:

Generators:

(1) $\mathbf{a}_i$ $(-t \le i \le t) : a_i^{-1} \to a_i$ glide reflections;

(2) $\mathbf{p}_i$ $(-t \le i \le t) : p_i^{-1} \to p_i$ screw motions (or translations);

(3) $\mathbf{r}_i (-t \le i \le t) : r_i^{-1} \to r_i$ screw motions (or translations);

(4) $\mathbf{s} := \mathbf{s}_u$ $(0 \le u \le t$ with fixed $u)$: $s_u^{-1} \to s_u$ a unique screw motion. Altogether: $3q + 1 = 6t + 4$ generators. That means, we consider only the "inner part" of Fig. 4.

Relations will be collected to arrowed (later to numbered) edges (not easily):

$$\text{e.g.} - \Delta\mathbf{a}_{-t}\mathbf{a}_{-t}\ldots\mathbf{a}_0\mathbf{a}_0\ldots\mathbf{a}_t\mathbf{a}_t = 1 \text{ identity, or } (\mathbf{a}_0\mathbf{p}_0\mathbf{a}_0^{-1}\mathbf{r}_1)3 = 1.$$

Think about (Fig. 4) the (first) starting vertical $-\Delta$ (arrowed) edge between faces a_1 and a_{-1}^{-1}. This will be mapped into the second $-\Delta$ between a_{-1}^{-1} and a_{-1} by mapping $\mathbf{a}_{-1} = \mathbf{a}_2$. Then follows again $\mathbf{a}_{-1}$, then $\mathbf{a}_0, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_1$ now with $6 = 2q$ ones in this edge class $-\Delta$. At each edge with angle $2 \times \pi/2q =: 2\zeta$ we get angle sum 2π indeed. Thus turning back to the (first) starting edge, we get the identity, also in general (this natural algorithm has a nice deep heuristic; please tile around the starting edge by corresponding inverse transforms as well)! In the above second relation to the arrowed edge class 3 we have 4 edges with rectangle at each: $4 \times \pi/2 = 2\pi$ indeed. This "cut-glue algorithm" also helps later on, as a "visual praxis" (not easy, of course), e.g. for fundamental domains of stabilizer subgroups.

But at the vertices $\bullet$, $\circ$, ending the edge class $\to$, we shall have angular neighbourhoods, amounting only a half ball at each. Thus, the equivalence classes C (Cu) and D (Cl) (you see them in Figs. 3, 7) are reflection centres with half ball neighbourhoods, i.e., punctures for quantum dots. The other point classes (orbits,

e.g. for $G, E, A, H, L, \ldots$ (in Figs. 3, 7) for occasional (Si) atoms) have ball neighbourhoods, as at manifolds.

The proportions of (Cu), (Cl) and (Si) can depend on parameter $q = 2t + 1$. We have infinitely many hyperbolic possibilities. The minimal one with $t = 1$, $q = 3$ is extremely interesting for (possible experimental) realisations, and shows the general rule as well!? For instance, it follows the presentation in case $q = 3$, $t = 1$, $u = 0$ in Figure 4 (in large freedom, of course, e.g. the natural order of relations can be more advantageous):

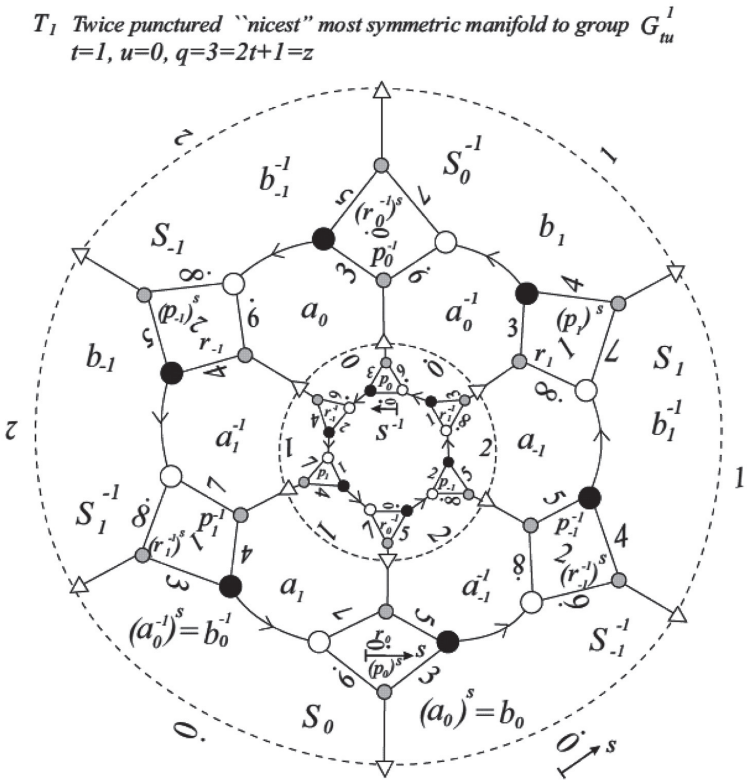


Figure 4. The "nicest" twice punctured (almost) manifold series T_1 with variable fundamental domains.

The basic presentation of its group $\mathbf{T}_1$ is:

$$\begin{aligned}
\mathbf{T}_1 = \{&\mathbf{a}_{-1}, \mathbf{a}_0, \mathbf{a}_1; \mathbf{s}_0 =: \mathbf{s}; \mathbf{p}_{-1}, \mathbf{p}_0, \mathbf{p}_1; \mathbf{r}_{-1}, \mathbf{r}_0, \mathbf{r}_1 - \\
&1\Delta = a_{-1}^2 \mathbf{a}_0^2 \mathbf{a}_1^2 \uparrow = (\mathbf{a}_{-1}\mathbf{s}\mathbf{a}_1\mathbf{s}^{-1})(\mathbf{a}_0\mathbf{s}\mathbf{a}_{-1}\mathbf{s}^{-1})(\mathbf{a}_1\mathbf{s}\mathbf{a}_0\mathbf{s}^{-1}) \\
&2 = (\mathbf{p}_{-1}\mathbf{s}\mathbf{r}_{-1}^{-1}\mathbf{s}) \;\; 0 = (\mathbf{p}_0\mathbf{s}\mathbf{r}_0^{-1}\mathbf{s}) \;\; 1 = (\mathbf{p}_1\mathbf{s}\mathbf{r}_1^{-1}\mathbf{s}) \\
&5 = (\mathbf{a}_{-1}\mathbf{p}_{-1}\mathbf{a}_{-1}^{-1}\mathbf{r}_0) \;\; 3 = (\mathbf{a}_0\mathbf{p}_0\mathbf{a}_0^{-1}\mathbf{r}_1) \;\; 4 = (\mathbf{a}_1\mathbf{p}_1\mathbf{a}_1^{-1}\mathbf{r}_{-1}) \\
&8 = (\mathbf{a}_{-1}\mathbf{r}_1^{-1}\mathbf{a}_{-1}^{-1}\mathbf{p}_{-1}^{-1}) \;\; 6 = (\mathbf{a}_0\mathbf{r}_{-1}^{-1}\mathbf{a}_0^{-1}\mathbf{p}_0^{-1}) \;\; 7 = (\mathbf{a}_1\mathbf{r}_0^{-1}\mathbf{a}_1^{-1}\mathbf{p}_1^{-1})\}
\end{aligned} \tag{2.8}$$

with trigonal rotational symmetric fundamental domain, (so with "mod 3 rules" in indices).

Remarks

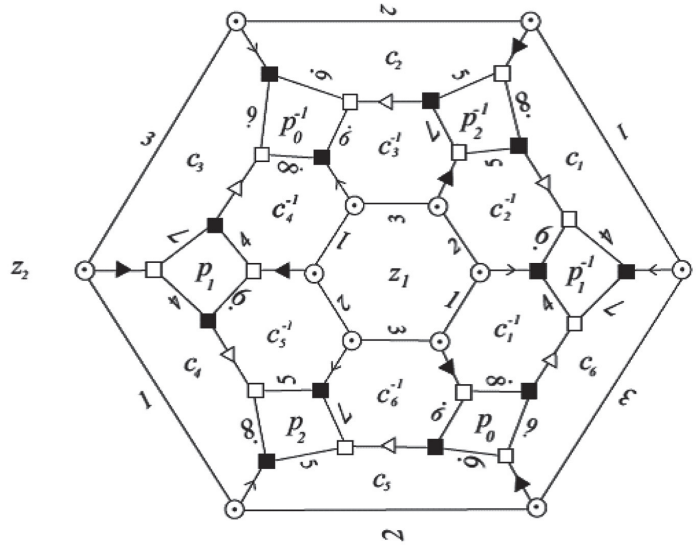
1. It was tedious work (by "cut-glue algorithm"), with lengthy computations in our original paper (Molnár, 1987) that both points C and D have fixing point reflections. Extra group computation was, how to derive this from the presentation of (try it on the above concrete example!).
2. By cutting-and-gluing the previous fundamental domain, e.g. by the screw motion $\mathbf{s}$, we can get other presentations, e.g. fewer generators and relations. Unifying the faces r_i and $(p_i)^{\mathbf{s}}$ to S_i and similarly p_i^{-1} and $(r_i)^{-1}$ to S_i^{-1}, we can introduce new generators $\mathbf{S}_i$: $S_i^{-1} \to S_i$ $(i = -1, 0, 1)$ with new elegant relations, according to the presentation (2.8). So we get a "minimal presentation" (with smallest number of generators and relations).
3. Between the dashed (- - -) circles we can get "the nicest fundamental domain"!!!
4. The cutting-gluing procedure serves us also the Dirichlet-Voronoi (D-V) cell of G (the midpoint of C and D punctures, e.g. in Figs. 3, 7). Then the punctures go into the middle of two opposite faces of the fundamental domain, as Figs. 5 a-b show to our both series T_1, T_2 (!?).
5. In our first series with screw motion generators $\mathbf{s}_u$, instead of $u = 0$; the parameter $u = 1 = t$ can lead us to non-isometric twice punctured (almost) manifold (also in general t, with t isometric pairs, also for our later manifolds.). This is not clear (not true!) for our second series with glide reflection generator $\mathbf{e}$.
6. In our Fig. 3 we indicated, how to derive in general the generators of fundamental groups of our tiling from the reflections of our truncated Coxeter simplex $\mathbf{C}_t$, so derive the complete presentation of the group as well. But this is tedious work.

As an amusing exercise, we derive the presentation of the above group to T_1 by our algorithmic procedure (to Fig. 5 a):

$$\begin{aligned}\mathbf{T}_1 = \{&\mathbf{z}_1, \mathbf{z}_2 \text{ point refl.; } \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5, \mathbf{c}_6 = \text{glide refl.; } \mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2 \text{ screw mot.} -\\ &1 = \mathbf{z}_1^2 = \mathbf{z}_2^2;\\ &\Delta = \mathbf{c}_1\mathbf{c}_2\mathbf{c}_3\mathbf{c}_4\mathbf{c}_5\mathbf{c}_6\uparrow = \mathbf{c}_1\mathbf{c}_6^{-1}\mathbf{c}_5\mathbf{c}_4^{-1}\mathbf{c}_3\mathbf{c}_2^{-1}\blacktriangleright = \mathbf{c}_1\mathbf{c}_2^{-1}\mathbf{c}_3\mathbf{c}_4^{-1}\mathbf{c}_5\mathbf{c}_6^{-1};\\ &1 = \mathbf{z}_1\mathbf{c}_1\mathbf{z}_2\mathbf{c}_4^{-1}\ \ 2 = \mathbf{z}_1\mathbf{c}_2\mathbf{z}_2\mathbf{c}_5^{-1}\ \ 3 = \mathbf{z}_1\mathbf{c}_3\mathbf{z}_2\mathbf{c}_6^{-1};\ \ 4 = \mathbf{c}_1\mathbf{p}_1\mathbf{c}_4\mathbf{p}_1^{-1}\\ &5 = \mathbf{c}_2\mathbf{p}_2\mathbf{c}_5\mathbf{p}_2^{-1}\ \ 6 = \mathbf{c}_3\mathbf{p}_0\mathbf{c}_6\mathbf{p}_0^{-1};\ \ 7 = \mathbf{p}_1\mathbf{c}_3^{-1}\mathbf{p}_2\mathbf{c}_6\ \ 8 = \mathbf{p}_2\mathbf{c}_4^{-1}\mathbf{p}_0\mathbf{c}_1\\ &9 = \mathbf{p}_0\mathbf{c}_5^{-1}\mathbf{p}_1^{-1}\mathbf{c}_2\}.\end{aligned} \tag{2.9}$$

You can similarly derive the presentation of $\mathbf{T}_2$ by Fig. 5 b.

Twice punctured manifold T_1,
$q=3=2t+1=z$



Twice punctured manifold T_2,

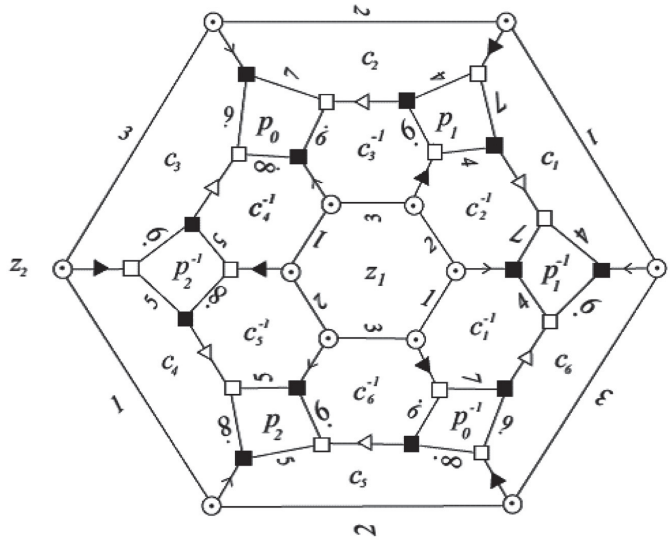


Figure 5. (a), (b) Note the two punctures at the middle of two opposite sides, as in Remark 4.

It is not clear yet whether these T_1 and T_2 are the same twice punctured tilings, as in Figs. 4 and 7, respectively.

2.2. Improving the original mistake: The first non-orientable manifold series to fundamental group series

We simply change the orientation preserving transforms $\mathbf{p}_i$ and $\mathbf{r}_i$ to orientation reversing ones, to $\mathbf{c}_i$ and $\mathbf{d}_i$, respectively $(i = -1, 0, 1)$; but also in general: $(-t \leq i \leq t)$. The previous orbits of C and D will be glued together, getting a ball neighbourhood of vertex class $\bullet$.

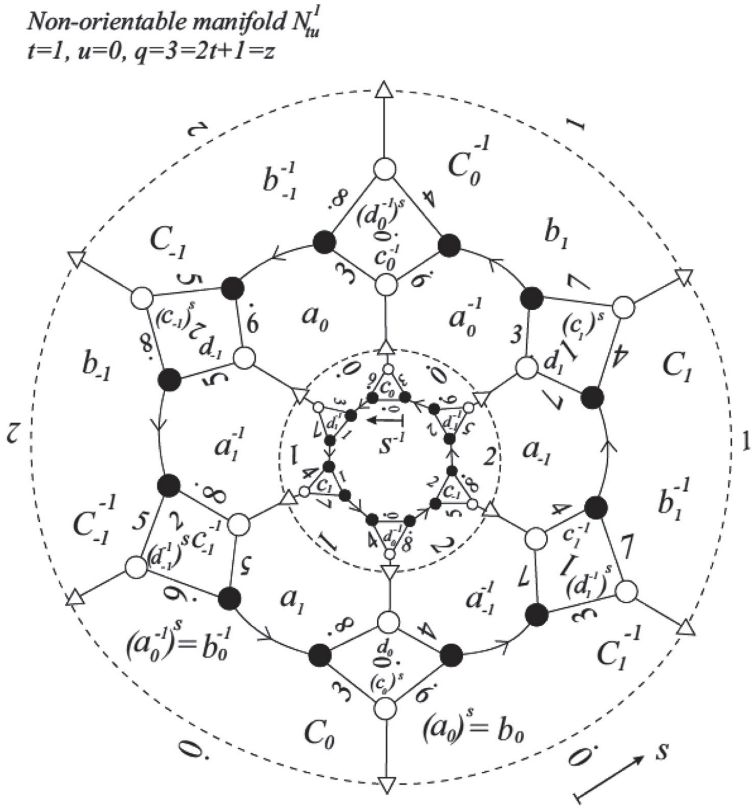


Figure 6. Our first non-orientable manifold series to groups by $t = 1, u = 0$, as before, with variable fundamental domains.

We illustrate our initiative: how to derive the ***minimal presentation*** of the above group $\mathbf{G}^1_{10}$ by gluing the face $(d_i^{-1})^{\mathbf{s}}$ to c_i^{-1} to get a new face C_i^{-1} and do $(c_i)^{\mathbf{s}}$ to d_i onto new C_i, to get a new generator $\mathbf{C}_i$: $C_i^{-1} \to C_i$ $(i = -1, 0, 1)$. We modify also faces s^{-1} and s, (by transform $\mathbf{s}$), of course:

$$\mathbf{G}^1_{10} = \{\mathbf{a}_{-1}, \mathbf{a}_0, \mathbf{a}_1 \text{ glide refl.}; \mathbf{C}_{-1}, \mathbf{C}_0, \mathbf{C}_1 \text{ glide refl.}; \mathbf{s} \text{ screw motion} -$$
$$1\Delta = \mathbf{a}^2_{-1}\mathbf{a}^2_0\mathbf{a}^2_1\uparrow = (\mathbf{a}_{-1}\mathbf{s}\mathbf{a}_1\mathbf{s}^{-1})(\mathbf{a}_0\mathbf{s}\mathbf{a}_{-1}\mathbf{s}^{-1})(\mathbf{a}_1\mathbf{s}\mathbf{a}_0\mathbf{s}^{-1});$$

$$
\begin{aligned}
3 &= \mathbf{a}_0\mathbf{C}_0 s^{-1}\mathbf{a}_0\mathbf{s}\mathbf{C}_1 \;\; 4 = \mathbf{a}_{-1}\mathbf{C}_1\mathbf{s}^{-1}\mathbf{a}_1\mathbf{s}\mathbf{C}_0 \\
5 &= \mathbf{a}_1\mathbf{C}_{-1}\mathbf{s}^{-1}\mathbf{a}_{.1}\mathbf{s}\mathbf{C}_{-1};\;\; 6 = \mathbf{a}_{-1}\mathbf{C}_0\mathbf{s}^{-1}\mathbf{a}_0^{-1}\mathbf{s}\mathbf{C}_{-1} \\
7 &= \mathbf{a}_{-1}^{.1}\mathbf{C}_1\mathbf{s}^{-1}\mathbf{a}_1^{.1}\mathbf{s}\mathbf{C}_1 \;\; 8 = \mathbf{a}_1^{-1}\mathbf{C}_{-1}\mathbf{s}^{-1}\mathbf{a}_{-1}^{-1}\mathbf{s}\mathbf{C}_0\}.
\end{aligned}
\tag{2.10}
$$

It is not easy to see that different $u - s$ (0 and 1) to the same t (now $t = 1$) lead to non-isometric manifolds.

3. Fundamental domains F_t^2 of groups $\mathbf{T}_2$ in $\mathbf{H}^3$ for analogous second series $\mathbf{T}_2$ of twice punctured (almost) manifolds in $\mathbf{H}^3$

We follow a completely analogous discussion, as before, now to Figs. 5 b, 7 much more briefly. The main difference is that the (say) bottom face e^{-1} is paired with the upper (infinite) face e by glide reflection **e**. The equivalence classes C (Cu) and D (Cl) are reflection centres ○, ●, respectively, in Fig. 7, with half-ball neighbourhoods, i.e., punctures for quantum dots.

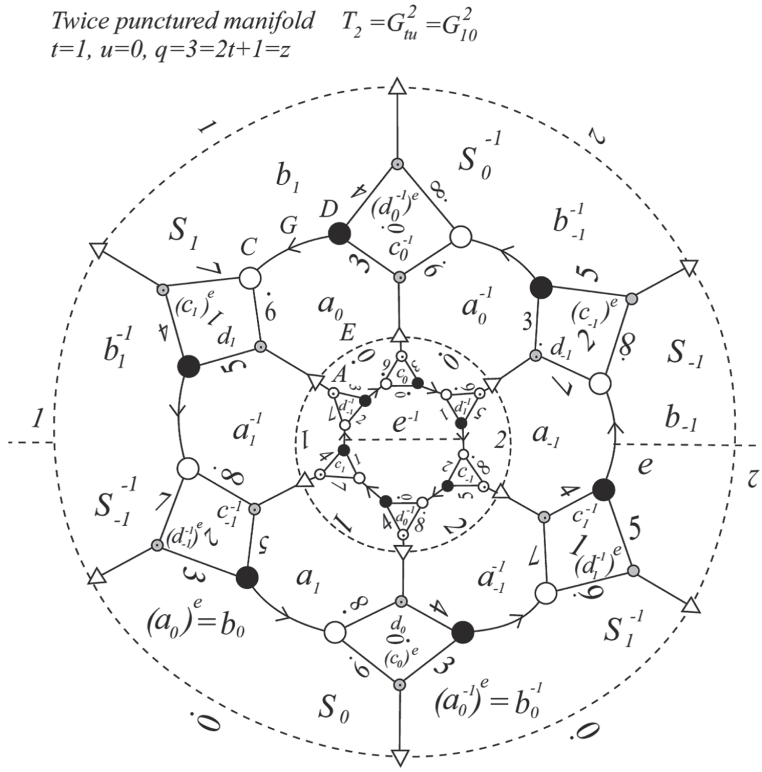


Figure 7. Our second twice punctured (almost) manifold series T_2 with the nicest domain between dashed - - - circles; with lower face e^{-1}, upper face e, paired by glide reflection **e** with a dashed glide plane.

We give the "nicest" presentation:

$$\begin{aligned}\mathbf{T}_2 = \{&\mathbf{a}_{-1},\mathbf{a}_0,\mathbf{a}_1;\mathbf{b}_{-1},\mathbf{b}_0,\mathbf{b}_1 \text{ glide refl.};\mathbf{e} \text{ glide refl.};\mathbf{S}_{-1},\mathbf{S}_0,\mathbf{S}_1 \text{ half-screw mot.}-\\ &1\uparrow = \mathbf{a}_0\mathbf{b}_1^{-1}\mathbf{a}_1\mathbf{b}_0^{-1}\mathbf{a}_{-1}\mathbf{b}_{-1}^{-1}\Delta = \mathbf{a}_0^2\mathbf{a}_1^2\mathbf{a}_{-1}^2\Delta = \mathbf{b}_0^2\mathbf{b}_1^2\mathbf{b}_{-1}^2;\\ &0 = \mathbf{a}_0\mathbf{e}\mathbf{b}_0^{-1}\mathbf{e}^{-1}\ 1 = \mathbf{a}_1\mathbf{e}\mathbf{b}_1^{-1}\mathbf{e}^{-1}\ 2 = \mathbf{a}_{-1}\mathbf{e}\mathbf{b}_{-1}^{-1}\mathbf{e}^{-1};\\ &3 = \mathbf{a}_0\mathbf{S}_0\mathbf{b}_0\mathbf{S}_{-1}\ 4 = \mathbf{a}_{-1}\mathbf{S}_1\mathbf{b}_1\mathbf{S}_0\ 5 = \mathbf{a}_1\mathbf{S}_{-1}\mathbf{b}_{-1}\mathbf{S}_1;\\ &6 = \mathbf{a}_0^{-1}\mathbf{S}_0\mathbf{b}_0^{-1}\mathbf{S}_1\ 7 = \mathbf{a}_{-1}^{-1}\mathbf{S}_1\mathbf{b}_1^{-1}\mathbf{S}_{-1}\ 8 = \mathbf{a}_1^{-1}\mathbf{S}_{-1}\mathbf{b}_{-1}^{-1}\mathbf{S}_0\}.\end{aligned} \tag{3.1}$$

In Fig. 5 b we have described another nice fundamental D-V-cell T_2 (?, but this is hard to prove!).

4. Our second series of non-orientable hyperbolic space forms to the previous domain F_t^2

Changing the generators $\mathbf{c}_i, \mathbf{d}_i$ to $\mathbf{p}_i, \mathbf{r}_i$, in Fig. 7, we glue together the former orbits of C and D to orbit $\bullet$ with ball neighbourhood in Fig. 8.

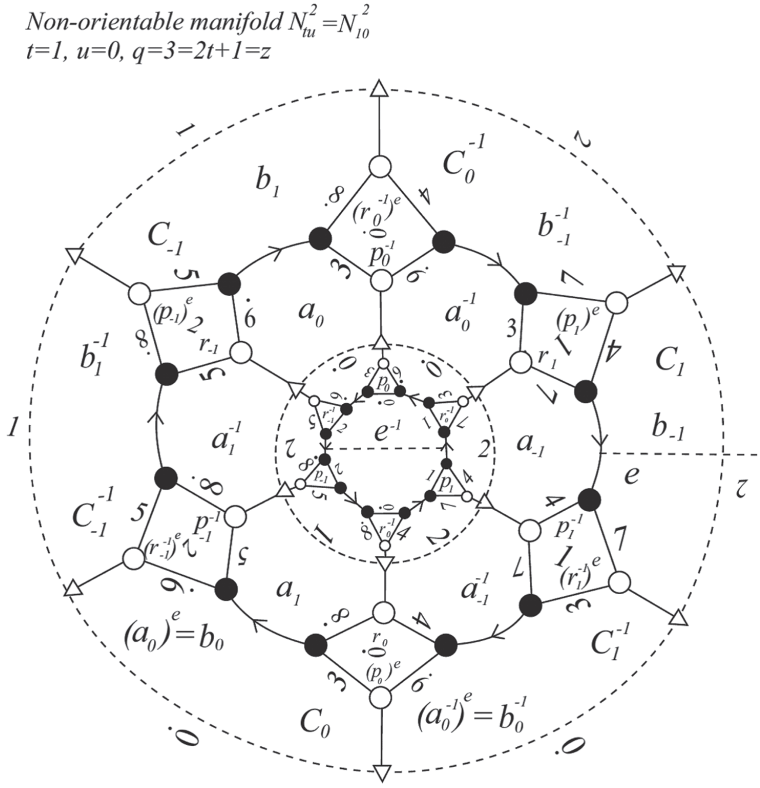


Figure 8. Our second non-orientable manifold series to groups indicated by $t = 1, u = 0$.

Here is the basic presentation (by the interior part of Fig. 8):

$$\begin{aligned}\mathbf{N}^2_{10} = \{&\mathbf{a}_{-1},\mathbf{a}_0,\mathbf{a}_1 \text{ glide refl.;}\mathbf{e} \text{ glide refl.; } \mathbf{p}_{-1},\mathbf{p}_0,\mathbf{p}_1;\mathbf{r}_{-1},\mathbf{r}_0,\mathbf{r}_1 \text{ hyp. transl.}-\\ &1\Delta = \mathbf{a}_0^2\mathbf{a}_1^2\mathbf{a}_{-1}^2\uparrow = (\mathbf{a}_0\mathbf{e}\mathbf{a}_1^{-1}\mathbf{e}^{-1})(\mathbf{a}_1\mathbf{e}\mathbf{a}_0^{-1}\mathbf{e}^{-1})(\mathbf{a}_{-1}\mathbf{e}\mathbf{a}_{-1}^{-1}\mathbf{e}^{-1});\\ &0 = \mathbf{p}_0\mathbf{e}\mathbf{r}_0^{-1}\mathbf{e}\ 1 = \mathbf{p}_1\mathbf{e}\mathbf{r}_1^{-1}\mathbf{e}\ 2 = \mathbf{p}_{-1}\mathbf{e}\mathbf{r}_{-1}^{-1}\mathbf{e};\\ &3 = \mathbf{a}_0\mathbf{p}_0\mathbf{a}_0^{-1}\mathbf{r}_1\ 4 = \mathbf{a}_{-1}\mathbf{p}_1\mathbf{a}_{-1}^{-1}\mathbf{r}_0\ 5 = \mathbf{a}_1\mathbf{p}_{-1}\mathbf{a}_1^{-1}\mathbf{r}_{-1};\\ &6 = \mathbf{a}_0^{-1}\mathbf{p}_0\mathbf{a}_0\mathbf{r}_{-1}\ 7 = \mathbf{a}_{-1}^{-1}\mathbf{p}_1\mathbf{a}_{-1}\mathbf{r}_1\ 8 = \mathbf{a}_1^{-1}\mathbf{p}_{-1}\mathbf{a}_1\mathbf{r}_0\}.\end{aligned} \tag{4.1}$$

For this series the isometry (the group isomorphism) problem seems to be hard, it is easy only for different parameters t, i.e. q by the Coxeter super groups $\mathbf{C}_t$ in Fig. 3.

5. Further plans

Our prismatic fundamental domain series derived specific Coxeter-Schläfli (doubly truncated) orthoschemes offer us further interesting analogous hyperbolic space groups to find. We plan to investigate them with Dr. Jenő SZIRMAI and our new Indonesian Doctor Colleague Arnasli YAHYA.

Acknowledgement

I thank my colleague Dr Jenő Szirmai for his essential help, especially for drawing the complicated nice figures.

Special thanks are due to the organisers.

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Contact address:
Emil Molnár
Department of Algebra and Geometry
Budapest University of Technology and Economics
Egry József utca 1, 1111 Budapest, Hungary
e-mail: emolnar@math.bme.hu

A “kvantum pöttyök” (Kémiai Nobel Díj 2023), mint a nem-irányítható tércsoportokról szóló (1983–84) hibás és javított cikkeim “alkalmazása”

Emil Molnár

Algebra és Geometria Tanszék, Budapesti Műszaki és Gazdaságtudományi Egyetem, Magyarország

Absztrakt. Közel 40 éves dolgozatom, mely a Bolyai Társulat Kollokvium sorozatának 46, 1984 évi Debrecen-Hajdúszoboszló konferencia-kötetében jelent meg 1987-ben, meglepően időszerűvé vált a 2023 évi Kémiai Nobel Díj Kitüntetettjeinek: Alekszej Jekimov, Louis E. Brus és Moungi G. Bawendi munkásságában. Természetesen, akkoriban nem sejthettem a téma időszerűségét, mely egy korábbi hibás cikkem következménye volt. Abban a dolgozatban új, nem-irányítható hiperbolikus térformák végtelen sorozatát akartam megszerkeszteni, mint Bolyai János $\mathbf{H}^3$ hiperbolikus geometriájának különleges poliéder-kitöltéseit. Szerencsére hamar észrevettem és kijavítottam a hibát. Ténylegesen, azok a konstrukciók nem voltak sokaságok, mert két fixpont-pálya szerepelt bennük, melyekben középpontos tükrözés lépett fel, mintegy “kilyukasztva” a sokaságot, ahol a pont “környezete” nem teljes labda, amit megkívánunk egy sokaságban, hanem a labda középpontra szimmetrikus pontjait “összeragasztjuk”. De ezek a különleges szinguláris pontok, mint “kvantum pöttyök” például réz és klór ionokkal üveg-, azaz szilícium-folyadékban (mely aztán “megfagy”) fényjelenséget okoznak elektronjaik “ugrabugrájával”, s a fény színe a kristályrészecskék méretétől függhet (gondoljunk csak a színes televízió képernyőjére).

Tehát a hiba sokkal érdekesebb volt, mint az eredeti szándék, melyet aztán hamar sikerült elérni!

Kulcsszavak: kvantum pöttyök, sokaság és kétszer lyukasztott (majdnem) sokaság, euklideszi és hiperbolikus kristálycsoport, poliéder kövezés

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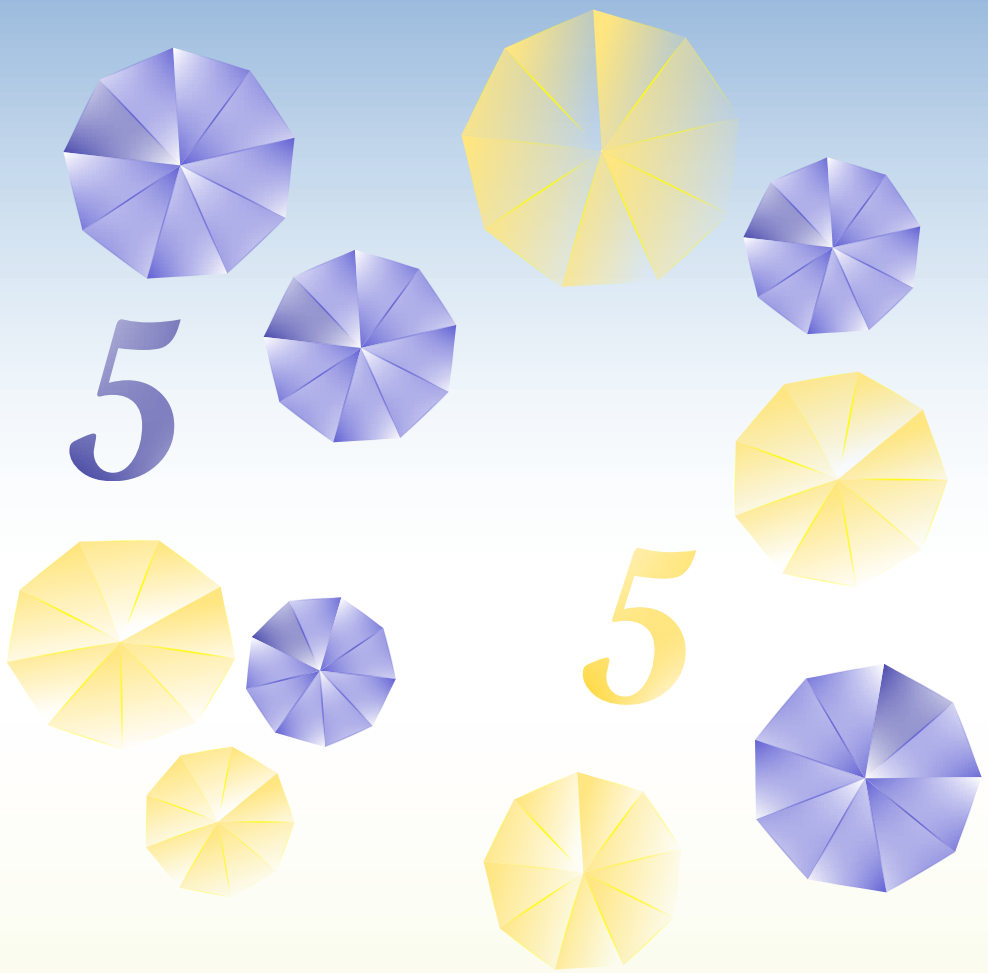

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